ÉQUATIONS AUX DÉRIVÉES PARTIELLES ELLIPTIQUES ET PARABOLIQUES SUR LES RÉSEAUX ET APPLICATIONS

Mémoire de Stage de M2

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Chapter 1

Introduction

There has been renew of interest for partial differential equations posed on networks in recent years, in particular in relation with Hamilton-Jacobi equations, initiated by the works of Y. Achdou, F. Camilli, A. Cutrì and N. Tchou [ACCT13] and C. Imbert, R. Monneau and H. Zidani [IMZ13]. However the study of partial differential equations on networks is a much older topic which can at least be traced back to the work of G. Lumer [Lum80a], [Lum80b] and [Lum84].

One of the recent developments, which was our main motivation for the present work, is the study of second order mean field games of networks initiated by F. Camilli and C. Marchi in [CM16] in the stationary case and continued by Y. Achdou, M.-K. Dao, O. Ley and N. Tchou in [ADLT19] for the stationary case and [ADLT20] for the dynamic case. Here the authors study the mean field game system of partial differential equations as introduced by J.-M. Lasry and P.-L. Lions in [LL07], [LL06a] and [LL06b]. See the survey by P. Cardaliaguet and A. Porretta [CP20] for a general introduction to the theory of mean field games.

One could say that three elements are needed in order to adapt the classical theory of mean field games to networks. Of course it is necessary to have a theory of elliptic and parabolic partial differential equations in order to write and solve the mean field game systems. But one also needs to be able to define stochastic processes as those are used to model the representative agent as well as a theory of optimal control in order to derive the system. In this work we are concerned with the first two aspects.

In Chapter 2 we give the main definitions and properties regarding networks and function spaces on networks which will be used in the rest of the text. In particular we prove that these function spaces enjoy the same properties as their usual analogues. These facts were previously used without proof in [CM16], [ADLT19] and [ADLT20].

The main chapter of this work is Chapter 3 where we study elliptic equations posed on networks. Our approach follows the one of [ADLT19] where we slightly changed the functional framework in order to simplify some of the proofs. In particular we were able to prove the existence and uniqueness result for weak solutions using the standard Lax-Milgram theorem while [ADLT19] relied on the less well-known Banach-Necas-Babuška theorem. This is achieved through the introduction of some well-chosen weighted spaces, following some ideas from [ADLT19]. We also systematically consider variable coefficients where [ADLT19] only considered constant second and zeroth order coefficients. Note that some of the result we obtain here can be seen as special cases of those proved by S. Nicaise in [Nic88] in the context of ramified spaces, which are generalization of networks to higher dimensions. This is the point of view originally adopted in [CM16]. We summarized the main results for this chapter in Fig. 3.1.

These results about elliptic equations are then applied in Chapter 4 to study linear parabolic equations on networks. We follow the semigroup method as presented in [Paz83] and [Hen81]. This approach is different from the one in [ADLT20] where the authors used the Galerkin method to prove the existence of weak solutions. At the time of writing we only considered autonomous problems but we believe that extensions to the non-autonomous case are possible. Again a similar approach was used by J. von Below and S. Nicaise for ramified spaces in [vBN96].

In Chapter 5 we deal with the question of stochastic processes on networks. The existence of diffusion processes on networks was stated by M. Freidlin and A. Wentzell in [FW93], where the proof is only sketched, and studied further by M. Freidlin and S.-J. Sheu in [FS00]. Here we provide a detailed proof for the existence of the process by applying the results on elliptic problems obtained in Chapter 3. This is a standard approach, see [Tai20] for instance. Next we prove that this Markov process admits an invariant measure and that this measure is unique and has a density. These facts were used without proof in [ADLT19].

Finally in Chapter 6 we make a formal derivation of the Hamilton-Jacobi-Bellman equation, assuming the usual optimal control theory holds in this context, and prove an existence and uniqueness result for stationary mean field game system with non-local coupling. This case was left as a remark in [ADLT19]. We conclude with an existence and uniqueness theorem for the parabolic Hamilton-Jacobi-Bellman equation using a standard theorem from the theory of semigroups.

I conclude this introduction by expressing my gratitude towards Olivier Ley and Francisco Silva for their supervision and advice in the preparation of this work. I also would like to thank Mériadec Chuberre and David Lassounon for always taking the time to answer my questions.

Chapter 2

Definitions and framework

This chapter presents the basic notions and properties that will be useful in the rest of the text. We first define (finite) networks and the relevant notions associated to it. We then introduce the main function spaces that we will need in the rest of the text. Our presentation closely follows the one in [ADLT19] and [ADLT20] which extends the one in [CM16]. We also provide proofs for the properties of the function spaces, especially for Sobolev spaces. These properties extend to networks most of what is true on an bounded interval of $\mathbb{R}$. To our knowledge the details of the proofs of these properties were never written. This can certainly be explained by the fact that these properties are very natural.

2.1 Networks

A *network* Γ is a connected subset of $\mathbb{R}^d$ made up of nodes linked by segments. More precisely consider $\mathcal{I}$ and $\mathcal{A}$ two subsets of $\mathbb{N}$ and let $\mathcal{V} = \{v_i \in \mathbb{R}^d : i \in \mathcal{I}\}$ be the set of *vertices* and $\mathcal{E} = \{\Gamma_\alpha \subset \mathbb{R}^d : \alpha \in \mathcal{A}\}$ be the set of *edges* where for each $\Gamma_\alpha \in \mathcal{E}$ there exists $v_i, v_j \in \mathcal{V}$, with $i \neq j$, such that $\Gamma_\alpha = \{\theta v_i + (1-\theta)v_j : \theta \in [0,1]\}$. We assume the the sets $\mathcal{I}$ and $\mathcal{A}$ are finite subsets of $\mathbb{N}$, which means that the network has a finite number of vertices and edges, and that for each pair $\alpha, \beta \in \mathcal{A}$, with $\alpha \neq \beta$, we have that $\Gamma_\alpha \cap \Gamma_\beta = \{v_i\}$ if there exists $i \in \mathcal{I}$ such that $\alpha, \beta \in \mathcal{A}_i$ and $\Gamma_\alpha \cap \Gamma_\beta = \varnothing$ otherwise. This last assumption means that edges may only intersect at a vertex. Finally we assume that every vertex belongs to an edge, to be precise for every $v_i \in \mathcal{V}$ there exists $\Gamma_\alpha \in \mathcal{E}$ such that $v_i \in \Gamma_\alpha$.

For $\alpha \in \mathcal{A}$, the *length* of the edge Γ_α will be denoted $\ell_\alpha \in (0,\infty)$ and Γ_α admits a parametrization $\pi_\alpha : [0, \ell_\alpha] \to \Gamma_\alpha$ defined by

$$\pi_\alpha(s) = \frac{1}{\ell_\alpha}((\ell_\alpha - s)v_i + sv_j). \tag{2.1}$$

Notice that π_α is an isometric homeomorphism from $[0, \ell_\alpha]$ to Γ_α.

Remark 2.1.1. *By splitting each edge of the network in two and adding an additional vertex at the new junction one may always consider a network for which either $\pi_\alpha^{-1}(v_i) = 0$ or $\pi_\alpha^{-1}(v_i) = \ell_\alpha$ for every $\alpha \in \mathcal{A}_i$ and each $i \in \mathcal{I}$. More precisely we can parameterize the network in such a way* *that $\pi_\alpha^{-1}(v_i) = 0$ if v_i is one of the original vertices and $\pi_\alpha^{-1}(v_i) = \ell_\alpha$ if it is one of the added ones, or conversely. This reparametrization is presented in Fig. 2.1.*

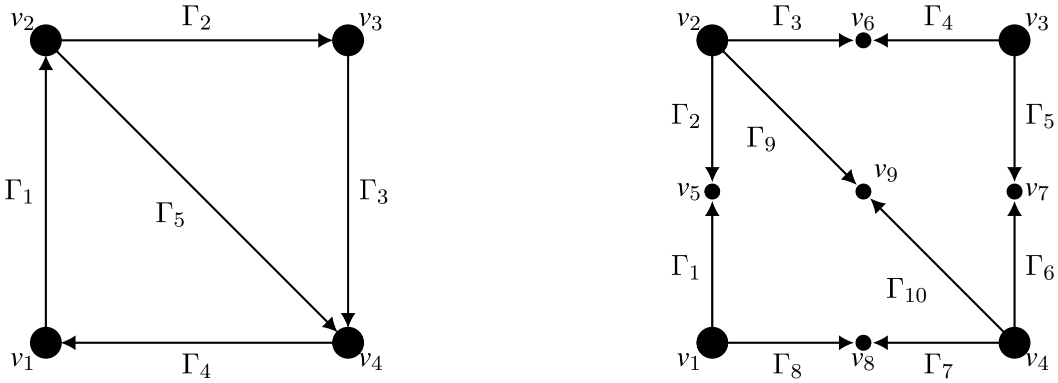

Figure 2.1 – The original network is presented on the left and the reparametrized one with added vertices on the right. The head of the arrow indicates the direction of the parametrization.

Definition 2.1.2. Let x and y be two points in Γ. By a *path* from x to y we mean an continuous map

$$[0, \ell] \ni s \mapsto \gamma(s) \in \Gamma$$

such that $\gamma(0) = x$ and $\gamma(\ell) = y$. Moreover for two vertices $v_i, v_j \in \mathcal{V}$, we say that $(\alpha_1, \ldots, \alpha_p) \in \mathcal{A}^p$ *induces a path* from v_i to v_j if the function γ defined by

$$[0, \ell] \ni s \mapsto \pi_{\alpha_k}\left(s - \sum_{n=1}^{k-1} \ell_{\alpha_n}\right) \quad \text{for } s \in \left[\sum_{n=1}^{k-1} \ell_{\alpha_n}, \sum_{n=1}^{k} \ell_{\alpha_n}\right]$$

defines a path from v_i to v_j. ◁

Remark 2.1.3. *Because Γ is assumed to be connected, for each pair $(i, j) \in \mathcal{I} \times \mathcal{I}$ there exists at least one finite sequence $(\alpha_1, \ldots, \alpha_p)$ that induces a path from v_i to v_j.*

Definition 2.1.4. We define the *shortest path* metric d on Γ by

$$d(x, y) = \begin{cases} |x - y| & \text{if } x, y \in \Gamma_\alpha \text{ for some } \alpha \in \mathcal{A}, \\ \inf_{\{\alpha_1, \ldots, \alpha_p\} \subset \mathcal{A}} \left\{ \sum_{i=2}^{p-1} \ell_{\alpha_i} + |v_{i(\alpha_1)} - x| + |v_{i(\alpha_p)} - y| \right\} & \text{otherwise ,} \end{cases}$$

where the infimum is taken over all the sequences $(\alpha_1, \ldots, \alpha_p)$ such that $x \in \Gamma_{\alpha_1}$ and $y \in \Gamma_{\alpha_p}$ with $\alpha_1 \in \mathcal{A}_{i(\alpha_1)}$, $\alpha_p \in \mathcal{A}_{i(\alpha_p)}$, $(\alpha_2, \ldots, \alpha_{p-1})$ which induces a path from $v_{i(\alpha_1)}$ to $v_{i(\alpha_p)}$ and $|\cdot|$ is the usual Euclidean norm on $\mathbb{R}^d$. ◁

Proposition 2.1.5. *The metric d is equivalent to the Euclidean metric on $\mathbb{R}^d$ restricted to Γ.*

Proof. See Proof A.1.1. □

Corollary 2.1.6. *The topology of* (Γ, d) *is equivalent to the topology induced by the Euclidean metric on* $\mathbb{R}^d$ *on* Γ *and the space* (Γ, d) *is thus a compact Polish space*[1].

To a function $u : \Gamma \to \mathbb{R}$ we associate for each $\alpha \in \mathcal{A}$ the function $u_\alpha : (0, \ell_\alpha) \to \mathbb{R}$ defined by

$$u_\alpha(y) = u \circ \pi_\alpha(y),$$

and when it is possible we define its values at the boundaries by

$$\begin{cases} u_\alpha(0) = \lim_{y \to 0^+} u_\alpha(y), \\ u_\alpha(\ell_\alpha) = \lim_{y \to \ell_\alpha^-} u_\alpha(y). \end{cases}$$

Finally, when the previous limits exist, we have

$$u_{|\Gamma_\alpha}(x) = \begin{cases} u_\alpha \circ \pi_\alpha^{-1}(x) & \text{for } x \in \Gamma_\alpha \setminus \mathcal{V}, \\ u_\alpha(0) & \text{if } x = v_i, \\ u_\alpha(\ell_\alpha) & \text{if } x = v_j, \end{cases}$$

for $x \in \Gamma_\alpha$ and $\Gamma_\alpha = [v_i, v_j]$.

We finish this section by defining the "Lebesgue" measure on Γ. Indeed Γ being a null measure set for the Lebesgue measure on $\mathbb{R}^d$, we cannot use this measure to define integrals on Γ. This can however easily be dealt with. We define the Lebesgue measure for every Borel set $A \in \mathcal{B}(\Gamma)$[2] by

$$\mathscr{L}(A) = \sum_{\alpha \in \mathcal{A}} \mathscr{L}_\alpha(\pi_\alpha^{-1}(A \cap \Gamma_\alpha))$$

where $\mathscr{L}_\alpha$ is the usual one dimensional Lebesgue measure on $[0, \ell_\alpha]$. Notice that π_α being a continuous function it is a $(\mathcal{B}([0, \ell_\alpha], \mathcal{B}(\Gamma))$-measurable function and therefore each $\mathscr{L}_\alpha \circ \pi_\alpha^{-1}$ defines a measure on Γ_α and $\mathscr{L}$ is a Borel measure on $(\Gamma, \mathcal{B}(\Gamma))$. Clearly it is a finite measure with total mass $\mathscr{L}(\Gamma) = \sum_{\alpha \in \mathcal{A}} \ell_\alpha$. For a $\mathscr{L}$-integrable function f we then have

$$\int_\Gamma f(x)\, \mathscr{L}(dx) = \sum_{\alpha \in \mathcal{A}} \int_{\Gamma_\alpha} f(x)\, \mathscr{L}_\alpha(dx) = \sum_{\alpha \in \mathcal{A}} \int_0^{\ell_\alpha} f \circ \pi_\alpha(y)\, dy.$$

In what follows we will simply write

$$\int_\Gamma f(x)\, dx := \int_\Gamma f(x)\, \mathscr{L}(dx).$$

Finally all the usual theorems about Lebesgue integrals hold for this measure, see [Bog07, Chapter 2].

[1] *I.e.* a complete and separable metric space.

[2] Note that as a consequence of Corollary 2.1.6 the Borel σ-algebra $\mathcal{B}(\Gamma)$ is the same if we consider the topology induced by the Euclidean metric of $\mathbb{R}^d$ or the one induced by the metric d.

2.2 Function spaces

In this section we define and give the main properties of the functions spaces that will be useful in the study of differential equations on networks. However we postpone the proofs of many of the results to Appendix A.2. The framework is highly inspired by the one used in [ADLT19], [ADLT20] and [CM16] where these function spaces are used but their properties are not proved.

As usual we denote $\mathscr{C}(\Gamma)$ the space of continuous real valued functions on Γ. Notice that as a consequence of Corollary 2.1.6 this space is the same for the topology of (Γ, d) and the topology induced by the Euclidean metric. It becomes a Banach space when equipped with the norm $\|u\|_{\mathscr{C}(\Gamma)} = \sup_{x\in\Gamma} |u(x)|$. In addition it will be convenient to allow functions to be discontinuous at the junctions but continuous on each edge. To this end we define the space

$$PC(\Gamma) = \left\{ u : \Gamma \to \mathbb{R} : \left| \begin{array}{l} u_\alpha \in \mathscr{C}(0, \ell_\alpha), \\ u_\alpha \text{ can be continuously extended to } [0, \ell_\alpha] \end{array} \right., \text{ for each } \alpha \in \mathcal{A} \right\}.$$

endowed with the norm of uniform convergence on each edge

$$\|u\|_{PC(\Gamma)} = \max_{\alpha\in\mathcal{A}} \|u_\alpha\|_\infty$$

which also makes it a Banach space. Notice that if $u \in PC(\Gamma)$ the we can define $u_{|\Gamma_\alpha}$ for each $\alpha \in \mathcal{A}$ and $u_{|\Gamma_\alpha} \in \mathscr{C}(\Gamma_\alpha)$, where $\mathscr{C}(\Gamma_\alpha)$ is the usual space of continuous real valued functions on Γ_α equipped with the topology induced by Γ. We clearly have the continuous embedding $\mathscr{C}(\Gamma) \hookrightarrow PC(\Gamma)$.

Let $u \in \mathscr{C}(\Gamma)$ and $\alpha \in \mathcal{A}$, then we can define the derivative of u at $x \in \Gamma_\alpha \setminus \mathcal{V}$ as a directional derivative. More precisely let $\Gamma_\alpha \in \mathcal{E}$ whose extremities are the vertices v_i and v_j. Define the unit vector

$$\boldsymbol{e}_\alpha = \frac{v_j - v_i}{|v_j - v_i|}.$$

Then the parametrization (2.1) can also be expressed

$$\pi_\alpha(s) = v_i + \frac{s}{\ell_\alpha}\boldsymbol{e}_\alpha \tag{2.2}$$

and in this case $\pi_\alpha(0) = v_i$ and $\pi_\alpha(\ell_\alpha) = v_j$. Then for a given x in the interior of Γ_α we can consider, when it exists, the directional derivative

$$D_\alpha u(x) = \lim_{h\to 0} \frac{u(x + h\boldsymbol{e}_\alpha) - u(x)}{h}.$$

Remark 2.2.1. *In order to make notations uniform, and following [ADLT19] and [ADLT20], in what follows we will use the notations ∂v for the usual derivative v' of a function defined and differentiable on an interval of $\mathbb{R}$.*

We see from (2.2) that $\pi_\alpha^{-1}(x) = x \cdot \boldsymbol{e}_\alpha$ and hence $(\pi_\alpha^{-1})'(x) = 1$ for every x in the interior of Γ_α. We now notice that

$$\begin{aligned}
\lim_{h\to 0} \frac{u(x+h\boldsymbol{e}_\alpha) - u(x)}{h} &= \lim_{h\to 0} \frac{u_\alpha(\pi_\alpha^{-1}(x+h\boldsymbol{e}_\alpha)) - u_\alpha(\pi_\alpha^{-1}(x))}{h} \\
&= \partial u_\alpha(\pi_\alpha^{-1}(x))(\pi_\alpha^{-1})'(x) \\
&= \partial u_\alpha(\pi_\alpha^{-1}(x)).
\end{aligned}$$

Hence in what follows we make the identification $D_\alpha u(x) = \partial u_\alpha(\pi_\alpha^{-1}(x))$, and we will write $\partial u(x) = \partial u_\alpha(\pi_\alpha^{-1}(x))$, but this is specific to the parametrization of the network we defined in (2.1) and (2.2). We define higher order derivatives in the same way and we write

$$\partial^k u(x) = \partial^k u_\alpha(\pi_\alpha^{-1}(x)) \quad \text{for } x \in \Gamma \setminus \mathcal{V}.$$

As we already mentioned above, the derivatives of u are in general not continuous at vertices. However we can always define $\partial_\alpha u(v_i)$ the *outward directional derivative* of u at $v_i \in \mathcal{V}$ for each $\alpha \in \mathcal{A}_i$ in the following way

$$\partial_\alpha u(v_i) = \begin{cases} \lim_{h\to 0^+} \frac{u_\alpha(0) - u_\alpha(h)}{h} & \text{if } v_i = \pi_\alpha(0), \\ \lim_{h\to 0^+} \frac{u_\alpha(\ell_\alpha) - u_\alpha(\ell_\alpha - h)}{h} & \text{if } v_i = \pi_\alpha(\ell_\alpha). \end{cases}$$

Notice that if we define

$$n_{i,\alpha} = \begin{cases} 1 & \text{if } v_i = \pi_\alpha(\ell_\alpha), \\ -1 & \text{if } v_i = \pi_\alpha(0), \end{cases} \tag{2.3}$$

then $\partial_\alpha u(v_i) = n_{i,\alpha} \partial u_\alpha(\pi_\alpha^{-1}(v_i))$.

We can thus define for every integer $k \geq 1$ the function space

$$\mathscr{C}^k(\Gamma) = \left\{ u \in \mathscr{C}(\Gamma) : \; u_\alpha \in \mathscr{C}^k([0,\ell_\alpha]), \; \forall \alpha \in \mathcal{A} \right\},$$

as well as

$$\mathscr{C}^\infty(\Gamma) = \left\{ u \in \mathscr{C}(\Gamma) : \; u \in \mathscr{C}^k(\Gamma), \; \forall k \geq 1 \right\}.$$

Remark 2.2.2. *Contrary to what one may expect we allow the derivatives of a function $u \in \mathscr{C}^k(\Gamma)$ to be discontinuous at the junctions.*

Proposition 2.2.3. *For each $k \geq 1$ the space $\mathscr{C}^k(\Gamma)$ equipped with the norm*

$$\|u\|_{\mathscr{C}^k(\Gamma)} = \sum_{\alpha \in \mathcal{A}} \sum_{0 \leq j \leq k} \|\partial^j u_\alpha\|_{L^\infty(0,\ell_\alpha)}$$

is a Banach space.

Proof. See Proof A.2.1. ☐

In what follows we will often need to extend the standard fact about the behavior of smooth functions at extremum points (*i.e.* the derivative cancels and the signed of the second derivative is prescribed) to the case where the extremum point is a vertex. Indeed the fact that the functions we consider are not smooth a the junctions imply that the usual result do not hold in general. However the *Kirchhoff condition* allows retrieve the usual properties.

Proposition 2.2.4. *Let $v_i \in \mathcal{V}$ and $u \in \mathscr{C}^2(\Gamma)$. Assume u satisfies the Kirchhoff condition*

$$\sum_{\alpha \in \mathcal{A}_i} p_{i,\alpha} \partial_\alpha u(v_i) = 0.$$

for some strictly positive real numbers $(p_{i,\alpha})_{\alpha \in \mathcal{A}_i}$. Suppose also that v_i is a local extremum of u. Then $\partial_\alpha u(v_i) = 0$ for every $\alpha \in \mathcal{A}_i$ and

$$\begin{cases} \partial^2 u_\alpha(\pi_\alpha^{-1}(v_i)) \leq 0 & \text{if } v_i \text{ is a local maximum,} \\ \partial^2 u_\alpha(\pi_\alpha^{-1}(v_i)) \geq 0 & \text{if } v_i \text{ is a local minimum.} \end{cases}$$

Proof. We only prove the case where v_i is a local maximum of u. The case of a minimum can be dealt with in a very similar fashion. We also assume that $v_i = \pi_\alpha(0)$ for every $\alpha \in \mathcal{A}_i$, this is always possible according to Remark 2.1.1. Again the argument is easily adapted to the case $v_i = \pi_\alpha(\ell_\alpha)$. Note also that the argument is reminiscent of [ADLT19, Lemma 3.3].

The fact that v_i is a local maximum of u implies that 0 is a local maximum of u_α for every $\alpha \in \mathcal{A}_i$ and thus

$$\partial_\alpha u(v_i) = \lim_{h \to 0^+} \frac{u_\alpha(0) - u_\alpha(h)}{h} \geq 0.$$

As each $p_{i,\alpha}$ is strictly positive the Kirchhoff condition then imposes $\partial_\alpha u(v_i) = 0$ for every $\alpha \in \mathcal{A}_i$. Now using the fact that $u_\alpha \in \mathscr{C}^2(\Gamma)$ we can write

$$u_\alpha(h) - u_\alpha(0) = \partial u_\alpha(0) h + \partial^2 u_\alpha(0) \frac{h^2}{2} + o(h^2).$$

Using the fact that $u_\alpha(h) - u_\alpha(0) \leq 0$ because 0 is a local maximum of u_α and as we just proved that $\partial u_\alpha(0) = 0$ we obtain

$$\partial^2 u_\alpha(0) \leq 0.$$

□

Remark 2.2.5. *The proof in fact shows that the conclusion of Proposition 2.2.4 still holds if we replace the Kirchhoff condition by*

$$\sum_{\alpha \in \mathcal{A}_i} p_{i,\alpha} \partial_\alpha u(v_i) \leq 0$$

for local maxima and

$$\sum_{\alpha \in \mathcal{A}_i} p_{i,\alpha} \partial_\alpha u(v_i) \geq 0$$

for local minima.

We will also need to consider *Hölder* (resp. *Lipschitz*) continuous functions on Γ. For each $k \in \mathbb{N}$ and $\theta \in (0,1)$ (resp. $\theta = 1$) we define

$$\mathscr{C}^{k,\theta}(\Gamma) = \left\{ u \in \mathscr{C}^k(\Gamma) : \ \|u\|_{\mathscr{C}^{k,\theta}(\Gamma)} < \infty \right\}$$

where

$$\|u\|_{\mathscr{C}^{k,\theta}(\Gamma)} = \|u\|_{\mathscr{C}^k(\Gamma)} + \max_{\alpha \in \mathcal{A}} \sup_{\substack{x,y \in [0,\ell_\alpha] \\ x \neq y}} \frac{|\partial^k u_\alpha(x) - \partial^k u_\alpha(y)|}{|x-y|^\theta}.$$

Proposition 2.2.6. *The space $\mathscr{C}^{k,\theta}(\Gamma)$ is a Banach space for each $k \in \mathbb{N}$ and every $\theta \in (0,1]$.*

Proof. The proof follows the same argument as for Proposition 2.2.3. □

Proposition 2.2.7. *There is a compact embedding from $\mathscr{C}^{0,\theta}(\Gamma)$ into $\mathscr{C}^{0,\gamma}(\Gamma)$ for $0 < \gamma < \theta \leq 1$ and for $u \in \mathscr{C}^{0,\theta}(\Gamma)$ we have*

$$\|u\|_{\mathscr{C}^{0,\gamma}(\Gamma)} \leq C \|u\|_{\mathscr{C}^{0,\theta}(\Gamma)}.$$

Proof. See Proof A.2.2. □

Having defined the Lebesgue measure on Γ in the previous section, we are able to define for $p \in [1,\infty]$ the *Lebesgue space* $L^p(\Gamma) = L^p(\Gamma, \mathscr{L})$ as usual (see [Bog07, Chapter 4] for instance). Notice that

$$L^p(\Gamma) = \{ u : \Gamma \to \mathbb{R} : \ u \in L^p(0,\ell_\alpha) \text{ for every } \alpha \in \mathcal{A} \}.$$

Moreover one can see that $\|u\|_{L^p(\Gamma)} = \left(\sum_{\alpha \in \mathcal{A}} \|u_\alpha\|^p_{L^p(0,\ell_\alpha)} \right)^{\frac{1}{p}}$ for $1 \leq p < \infty$ and $\|u\|_{L^\infty(\Gamma)} = \max_{\alpha \in \mathcal{A}} \|u_\alpha\|_{L^\infty(0,\ell_\alpha)}$.

Proposition 2.2.8. *For every $p \in [1,\infty]$ the space $L^p(\Gamma)$ is a Banach space and $L^2(\Gamma)$ is an Hilbert space for the scalar product*

$$(u,v)_{L^2(\Gamma)} = \int_\Gamma u(x)v(x)\, dx.$$

Proof. This can be proved in two ways. Either one can use an argument similar to the one used in Proposition 2.2.3 by coming back to usual Lebesgue spaces on an interval or one can directly use the result for abstract measures, see [Bog07, Theorem 4.1.3]. □

Finally we will also need to work with weakly-differentiable functions on Γ.

Definition 2.2.9. Let $u : \Gamma \to \mathbb{R}$ be a function. Assume that for every $\alpha \in \mathcal{A}$ the usual weak derivative of u_α is a function on $(0,\ell_\alpha)$. Then we define the *weak derivative* ∂u of u by

$$\partial u(x) = \partial u_\alpha(\pi_\alpha^{-1}(x)) \quad \text{for } x \in \Gamma_\alpha \setminus \mathcal{V}.$$

◁

This leads us to define *Sobolev spaces* on Γ.

Definition 2.2.10. For any integer $k \geq 1$ and every $p \in [1, \infty]$ we define the Sobolev space

$$W^{k,p}(\Gamma) = \left\{ u \in \mathscr{C}(\Gamma) : \|u\|_{W^{k,p}(\Gamma)} < \infty \right\}$$

where

$$\|u\|_{W^{k,p}(\Gamma)} = \left(\|u\|^p_{L^p(\Gamma)} + \sum_{j=1}^{k} \|\partial^j u\|^p_{L^p(\Gamma)} \right)^{\frac{1}{p}} \quad \text{for } 1 \leq p < \infty,$$

and

$$\|u\|_{W^{k,\infty}(\Gamma)} = \|u\|_{L^\infty(\Gamma)} + \sum_{j=1}^{k} \|\partial^j u\|_{L^\infty(\Gamma)}.$$

Occasionally it will be convenient to drop the continuity assumption of the functions at vertices, therefore we also define the *"broken" Sobolev spaces*

$$W^{k,p}_b(\Gamma) = \left\{ u \in L^p(\Gamma) : \; u_\alpha \in W^{k,p}([0, \ell_\alpha]), \; \forall \alpha \in \mathcal{A} \right\}.$$

$\triangleleft$

Remark 2.2.11. *Notice that a function $u \in \mathscr{C}(\Gamma)$ belongs to $W^{k,p}(\Gamma)$ if, and only if, it verifies $u_\alpha \in W^{k,p}(0, \ell_\alpha)$ for every $\alpha \in \mathcal{A}$.*

As usual we will denote $H^k(\Gamma) = W^{k,2}(\Gamma)$. In the rest of this section we verify that the spaces $W^{k,p}(\Gamma)$ satisfy the main properties we expect from a function space named after Sergueï L. Sobolev. Most of the proofs rely on the properties satisfied by usual one dimensional Sobolev spaces, see [Bre11, Chapter 8].

Proposition 2.2.12. *For any $k \geq 1$ and every $p \in [1, \infty]$, the space $W^{k,p}(\Gamma)$ is a Banach space and $H^k(\Gamma)$ is an Hilbert space.*

Proof. See Proof A.2.3. $\square$

Proposition 2.2.13. *There is a continuous injection $W^{1,1}(\Gamma) \hookrightarrow L^q(\Gamma)$ for every $q \in [1, \infty]$ and the injection is compact for $1 \leq q < \infty$.*

Proof. See Proof A.2.5. $\square$

Proposition 2.2.14. *The Sobolev space $W^{1,p}(\Gamma)$ is a compact subset of $\mathscr{C}(\Gamma)$ for $1 < p \leq \infty$.*

Proof. This is the same argument as in Proposition 2.2.13 using the compact embedding

$$W^{1,p}(0, \ell_\alpha) \hookrightarrow \mathscr{C}([0, \ell_\alpha]),$$

see [Bre11, Theorem 8.8]. $\square$

Proposition 2.2.15 (Morrey's inequality). *Let* $u \in W^{1,p}(\Gamma)$ *for* $1 < p \leq \infty$. *Then* $u \in \mathscr{C}^{0,\theta}(\Gamma)$ *with* $0 < \theta \leq 1 - \frac{1}{p}$ *and*

$$\|u\|_{\mathscr{C}^{0,\theta}(\Gamma)} \leq C\|u\|_{W^{1,p}(\Gamma)}.$$

Furthermore, the embedding is compact for $0 < \theta < 1 - \frac{1}{p}$.

Proof. See Proof A.2.6. □

Chapter 3

Second order linear elliptic equations

This section is dedicated to the study of a general class of linear elliptic partial differential equations posed on networks. The class of equations which we study is similar to the one that is considered in the corresponding chapter of [Eva10]. Our goal is to generalize results presented in [ADLT19] and [CM16] which also greatly influenced the presentation. In [CM16] special cases of the results we present here are used without proof and the authors refer to [Nic88] where one can indeed find result similar to ours regarding weak solutions but for more general ramified spaces. In [ADLT19] the authors provide proofs specialized on networks but only for second and zeroth order terms which are constant on each edge. We extend here this approach to variable coefficients. We were also able to slightly simplify some of the proofs by a change of function space.

Note that technically the problem consists of a coupled system of differential equations posed on intervals. As everything is one dimensional the name partial differential equation is not very accurate however the tools we use in the study of the problem clearly come from partial differential equations which is the reason for this choice of terminology. To fix notations we consider the differential operator L defined by

$$Lu(x) = -a(x)\partial^2 u(x) + b(x)\partial u(x) + c(x)u(x) \quad \text{for } x \in \Gamma \setminus \mathcal{V}, \tag{3.1}$$

where we assume at least $a, b, c \in L^\infty(\Gamma)$. Moreover for every $\alpha \in \mathcal{A}$ we define the differential operator L_α by

$$L_\alpha v(y) = -a_\alpha(y)\partial^2 v(y) + b_\alpha(y)\partial v(y) + c_\alpha(y)v(y) \quad \text{for } y \in (0, \ell_\alpha), \tag{3.2}$$

for any function $v \in \mathscr{C}^2(0, \ell_\alpha)$. The operator L will be called *(uniformly) elliptic* if there exists a positive constant ω such that $a(x) \geq \omega > 0$ for every $x \in \Gamma \setminus \mathcal{V}$.

The main results obtained in this section are summarized in Fig. 3.1.

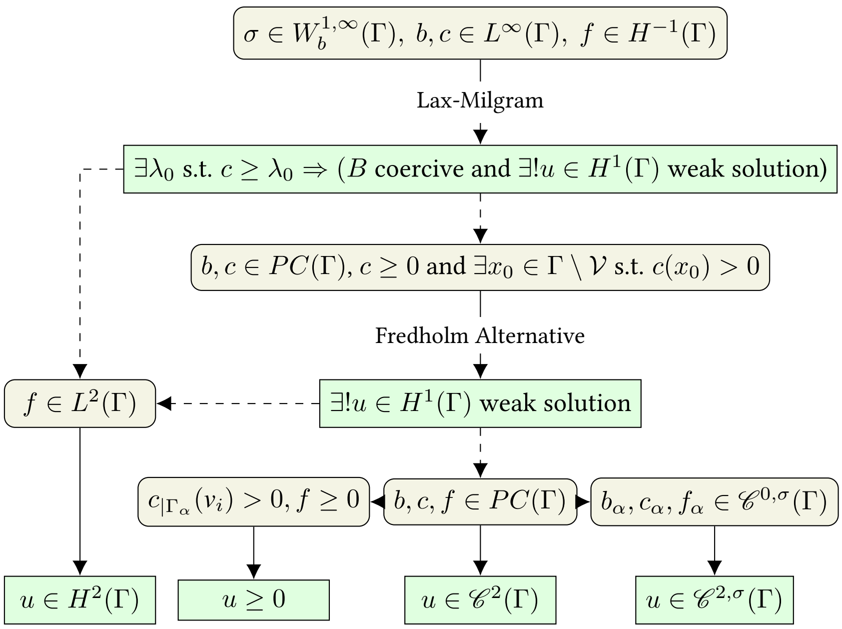

Figure 3.1 – Linear elliptic theory on networks. Assumptions are presented in grey boxes and results in green ones.

3.1 Maximum principles

In this section we extend the standard maximum principles for elliptic equations on an interval. Our presentation in strongly influenced by the ones in [Eva10] and [GT01].

Theorem 3.1.1 (Weak maximum principle)**.** *Consider a function $u \in \mathscr{C}^2(\Gamma)$ and assume $c = 0$ on Γ.*

1. *Suppose*
$$Lu \leq 0 \text{ on } \Gamma \setminus \mathcal{V},$$
then we have
$$\max_{x \in \Gamma} u(x) = \max_{x \in \mathcal{V}} u(x).$$

2. *Suppose*
$$Lu \geq 0 \text{ on } \Gamma \setminus \mathcal{V},$$
then we have
$$\min_{x \in \Gamma} u(x) = \min_{x \in \mathcal{V}} u(x).$$

Proof. See Proof A.3.1. □

Corollary 3.1.2. *Consider a function $u \in \mathscr{C}^2(\Gamma)$ and assume $c \geq 0$ on Γ.*

1. *Suppose*
$$Lu \leq 0 \text{ in } \Gamma \setminus \mathcal{V},$$
then we have
$$\max_{x\in\Gamma} u(x) \leq \max_{x\in\mathcal{V}} u^+(x).$$
2. *Suppose*
$$Lu \geq 0 \text{ in } \Gamma \setminus \mathcal{V},$$
then we have
$$\min_{x\in\Gamma} u(x) \geq \min_{x\in\mathcal{V}} u^-(x).$$

Proof. Use the same argument as in Theorem 3.1.1, using the analogue result in intervals ([Eva10, Theorem 2 p.348], [GT01, Corollary 3.2]). □

Theorem 3.1.3. *Assume $c \geq 0$. Let $u \in \mathscr{C}^2(\Gamma)$ be such that*
$$Lu = 0$$
and satisfies the Kirchhoff condition
$$\sum_{\alpha\in\mathcal{A}_i} p_{i,\alpha}\partial_\alpha u(v_i) = 0 \quad \forall i \in \mathcal{I}$$
where $(p_{i,\alpha})_{\alpha\in\mathcal{A}_i}$ is a family of positive real numbers for each $i \in \mathcal{I}$. Then u is constant. Furthermore if there exists some $x_0 \in \Gamma \setminus \mathcal{V}$ such $c(x_0) > 0$, then $u = 0$.

Proof. First notice that if we replace u by $-u$ the function stills satisfies the assumptions of the theorem. Therefore, by the weak maximum principle 3.1.2, we may assume that u achieves a nonnegative maximum at some vertex $v_i \in \mathcal{V}$. We now claim that $u_{|\Gamma_\alpha}$ is constant for every $\alpha \in \mathcal{A}_i$. Indeed if it is not the case for some $\alpha \in \mathcal{A}_i$, then as $L_\alpha u_\alpha = 0$ we may apply the usual strong maximum principle (see [GT01, Theorem 3.5]) to deduce that $u_{|\Gamma_\alpha}(v_i) > u_{|\Gamma_\alpha}(x)$ for every x in the interior of Γ_α. Then Hopf's lemma (see [GT01, Lemma 3.4]) implies $\partial_\alpha u(v_i) > 0$ and contradicts Proposition 2.2.4 which states that $\partial_\alpha u(v_i) = 0$. Hence $u_{|\Gamma_\alpha}$ must be constant for every $\alpha \in \mathcal{A}_i$ and the constants are identical by continuity of u on Γ. Now for every $j \in \mathcal{I}$ such that $v_j \in \Gamma_\alpha$ for some $\alpha \in \mathcal{A}_i$ we have $u(v_j) = \max_{x\in\Gamma} u(x)$ and therefore we can propagate the argument across the network.

The final statement comes from the fact that zero is the only constant satisfying $Lu = 0$ when $c(x_0) > 0$ for some $x_0 \in \Gamma \setminus \mathcal{V}$. □

Remark 3.1.4. *In the case where $c = 0$ a more elementary proof of Theorem 3.1.3 is given in [ADLT19, Lemma 2.5].*

3.2 Problems with Kirchhoff transmission condition

Let for each $i \in \mathcal{I}$ let $(\gamma_{i,\alpha})_{\alpha\in\mathcal{A}}$ be positive real numbers. In this section we study the well-posedness of the following problem

$$(\mathscr{E}_K) \qquad \begin{cases} -\sigma^2(x)\partial^2 u(x) + b(x)\partial u(x) + c(x)u(x) = f(x) & x \in \Gamma \setminus \mathcal{V}, \\ \sum_{\alpha\in\mathcal{A}_i} \gamma_{i,\alpha}\sigma_{|\Gamma_\alpha}(\nu_i)\partial_\alpha u(\nu_i) = 0 & \forall i \in \mathcal{I}, \\ u_{|\Gamma_\alpha}(\nu_i) = u_{|\Gamma_\beta}(\nu_i) & \forall \alpha, \beta \in \mathcal{A}_i,\ i \in \mathcal{I}. \end{cases}$$

The condition in the second line is called the *Kirchhoff transmission condition.*

We begin with a preliminary remark. Consider $u, v \in \mathscr{C}^2(\Gamma)$ and we compute using integration by parts

$$\begin{aligned} (-\partial^2 u, v)_{L^2(\Gamma)} &= \int_\Gamma -\partial^2 u(x)v(x)\,dx = \sum_{\alpha\in\mathcal{A}} \int_0^{\ell_\alpha} -\partial^2 u_\alpha(x)v_\alpha(x)\,dx \\ &= \sum_{\alpha\in\mathcal{A}} \int_0^{\ell_\alpha} \partial u_\alpha(x)\partial v_\alpha(x)\,dx - \sum_{i\in\mathcal{I}} \sum_{\alpha\in\mathcal{A}_i} [\partial u_\alpha(x)v_\alpha(x)]_0^{\ell_\alpha} \\ &= \int_\Gamma \partial u(x)\partial v(x)\,dx + \sum_{i\in\mathcal{I}} v(\nu_i) \sum_{\alpha\in\mathcal{A}_i} \partial_\alpha u(\nu_i). \end{aligned}$$

Notice that the Kirchhoff condition almost appears on the last line. In order to obtain it we need a way to make the coefficients $(\gamma_{i,\alpha})_{\alpha\in\mathcal{A}_i}$ appear in the computation. This will be achieved with the following function.

Definition 3.2.1. Let the functions $\psi \in PC(\Gamma)$ be defined as follows :

$$\begin{cases} \psi_\alpha \text{ is affine on } (0, \ell_\alpha), \\ \psi_{|\Gamma_\alpha}(\nu_i) = \gamma_{i,\alpha} \text{ if } \alpha \in \mathcal{A}_i. \end{cases}$$

◁

Note that the function ψ is positive, bounded and only depends on the Kirchhoff condition. Then we obtain with the same computation

$$(\partial^2 u, v\psi)_{L^2(\Gamma)} = \int_\Gamma \partial u(x)\partial(v(x)\psi(x))\,dx + \sum_{i\in\mathcal{I}} v(\nu_i) \sum_{\alpha\in\mathcal{A}_i} \gamma_{i,\alpha}\partial_\alpha u(\nu_i).$$

This leads us to consider the following *weighted spaces*

$$L^p(\Gamma;\psi) = \left\{ u : \Gamma \to \mathbb{R} :\ \|u\|_{L^p(\Gamma;\psi)} = \left(\int_\gamma |u(x)|^p \psi(x)\,dx \right)^{\frac{1}{p}} < \infty \right\}$$

and the space $L^2(\Gamma;\psi)$ is naturally endowed with the following inner product

$$(u, v)_{L^2(\Gamma;\psi)} = \int_\Gamma u(x)v(x)\psi(x)\,dx.$$

Because ψ is a positive bounded function we in fact have $L^p(\Gamma;\psi) = L^p(\Gamma)$, the norms are equivalent and we denote c_ψ and C_ψ two positive constants such that

$$c_\psi \|u\|_{L^p(\Gamma;\psi)} \leq \|u\|_{L^p(\Gamma)} \leq C_\psi \|u\|_{L^p(\Gamma;\psi)}.$$

In particular $L^p(\Gamma;\psi)$ are Banach spaces for $1 \leq p \leq \infty$ and $L^2(\Gamma;\psi)$ is an Hilbert space. Analogously we can define the *weighted Sobolev spaces* $W^{k,p}(\Gamma;\psi)$ which are the spaces $W^{k,p}(\Gamma)$ using norms in $L^p(\Gamma;\psi)$ instead of $L^p(\Gamma)$ for the weak derivatives. In particular $H^k(\Gamma;\psi)$ is an Hilbert space provided with the inner product

$$(u,v)_{H^k(\Gamma;\psi)} = \sum_{0\leq l\leq k} (\partial^l u, \partial^l v)_{L^2(\Gamma;\psi)}.$$

Definition 3.2.2. A *classical solution* of ($\mathscr{E}_K$) is a function $u \in \mathscr{C}^2(\Gamma)$ verifying

$$Lu(x) = f(x) \quad \forall x \in \Gamma \setminus \mathcal{V}$$

pointwise and such that

$$\sum_{\alpha\in\mathcal{A}_i} \gamma_{i,\alpha} \sigma^2_{|\Gamma_\alpha}(v_i) \partial_\alpha u(v_i) = 0 \quad \forall i \in \mathcal{I}.$$

◁

We now derive the weak formulation of ($\mathscr{E}_K$). In what follows it will be convenient to write the problem in divergence form. Hence, assuming σ is regular enough we write

$$Lu(x) = -\partial(a(x)\partial u(x)) + \tilde{b}(x)\partial u(x) + c(x)u(x) \quad \text{for every } x \in \Gamma \setminus \mathcal{V}.$$

where $a = \sigma^2$ and $\tilde{b} = b + 2\sigma\partial\sigma$. Let u be a classical solution of ($\mathscr{E}_K$), multiplying by $v\psi$, with $v \in \mathscr{C}^2(\Gamma)$, in ($\mathscr{E}_K$) and integrating by parts we get

$$\begin{aligned}
(f,v)_{L^2(\Gamma;\psi)} = (Lu,v)_{L^2(\Gamma;\psi)} &= \int_\Gamma -\partial(a\partial u)v\psi + \tilde{b}\partial u v\psi + cuv\psi \, dx \\
&= \int_\Gamma a\partial u \partial(v\psi) + \tilde{b}\partial u v\psi + cuv\psi \, dx + \sum_{i\in\mathcal{I}} v(v_i) \sum_{\alpha\in\mathcal{A}_i} \gamma_{i,\alpha}\sigma^2_{|\Gamma_\alpha}(v_i)\partial_\alpha u(v_i) \\
&= \int_\Gamma a\partial u \partial(v\psi) + \tilde{b}\partial u v\psi + cuv\psi \, dx
\end{aligned}$$

where we used the fact that u satisfies the Kirchhoff condition to obtain the last line. Hence u solves

$$B(u,v) = (f,v)_{L^2(\Gamma;\psi)} \quad \text{for every } v \in H^1(\Gamma), \tag{3.3}$$

where B is the bilinear form defined on $H^1(\Gamma) \times H^1(\Gamma)$ by

$$B(u,v) = \int_\Gamma a\partial u \partial(v\psi) + \tilde{b}\partial u v\psi + cuv\psi dx. \tag{3.4}$$

Conversely assume that $u \in \mathscr{C}^2(\Gamma)$ solves (3.3). Then going backward in the computation above gives

$$(f, v)_{L^2(\Gamma;\psi)} = \int_\Gamma -\partial(a\partial u)v\psi + \tilde{b}\partial uv\psi + cuv\psi \, dx - \sum_{i \in \mathcal{I}} v(\nu_i) \sum_{\alpha \in \mathcal{A}_i} \gamma_{i,\alpha}\sigma^2_{|\Gamma_\alpha}(\nu_i)\partial_\alpha u(\nu_i).$$

Using test functions $v \in H^1(\Gamma)$ that are compactly supported in Γ_α we deduce that

$$-\sigma^2(x)\partial^2 u(x) + b(x)\partial u(x) + c(x)u(x) = f(x) \quad \text{for almost every } x \in \Gamma_\alpha \setminus \mathcal{V}$$

and the regularity of u implies that it is in fact the case everywhere on $\Gamma_\alpha \setminus \mathcal{V}$, this is the first line in ($\mathscr{E}_K$). Next for $\epsilon > 0$ small enough and $i \in \mathcal{I}$ we can choose test functions $v^i_\epsilon \in H^1(\Gamma)$ of the following form :

$$\begin{cases} v^i_{\epsilon,\alpha} \text{ is piecewise affine on } [0, \ell_\alpha], \\ v^i_\epsilon(\nu_j) = \delta_{i,j}, \\ \operatorname{supp} v^i_\epsilon \subset \mathcal{B}(\nu_i, \epsilon). \end{cases}$$

Notice that they can be chosen to lie in a bounded subset of $L^\infty(\Gamma;\psi)$ and an application of the dominated convergence theorem gives

$$\lim_{\epsilon \to 0} (f, v^i_\epsilon)_{L^2(\Gamma;\psi)} = 0,$$

$$\lim_{\epsilon \to 0} \int_\Gamma (-\partial(a\partial u) + \tilde{b}\partial u + cu)v^i_\epsilon \psi \, dx = 0$$

and thus we obtain

$$0 = \sum_{\alpha \in \mathcal{A}_i} \gamma_{i,\alpha}\sigma^2_{|\Gamma_\alpha}(\nu_i)\partial_\alpha u(\nu_i),$$

when ϵ tends to 0 which is the second line in ($\mathscr{E}_K$). Finally the third line of ($\mathscr{E}_K$) is implied by $u \in H^1(\Gamma)$. This shows that u is then a solution of ($\mathscr{E}_K$).

We are able to state the general weak formulation of ($\mathscr{E}_K$) :

$$(\mathscr{E}') \qquad \left| \begin{array}{l} \text{Given } f \in H^{-1}(\Gamma) \text{ find } u \in H^1(\Gamma) \text{ such that} \\ B(u, v) = \langle f, v \rangle_{H^{-1},H^1} \quad \text{for every } v \in H^1(\Gamma), \end{array} \right.$$

where B is the bilinear form defined in (3.4) and $H^{-1}(\Gamma)$ is the dual space of $H^1(\Gamma)$. A function satisfying ($\mathscr{E}'$) is called a *weak solution* of ($\mathscr{E}_K$).

In order to solve ($\mathscr{E}'$) we make the following assumptions on the coefficients σ, b and c

$$(H_1) \qquad 0 < \omega \leq \inf_{x \in \Gamma \setminus \mathcal{V}} a(x) = \inf_{x \in \Gamma \setminus \mathcal{V}} \sigma^2(x)$$

for some constant ω,

$$(H_2) \qquad b, c \in L^\infty(\Gamma), \ \sigma \in W^{1,\infty}(\Gamma_\alpha) \text{ for every } \alpha \in \mathcal{A}.$$

And finally

$$(H_3) \qquad 0 < \lambda \leq \inf_{x \in \Gamma \setminus \mathcal{V}} c(x),$$

for some constant λ.

The following result generalizes the first step in the proof of [ADLT19, Lemma 2.3]. This can also be seen as a special case of [Nic88, Theorem 2.2] proved in the context of ramified spaces.

Theorem 3.2.3. *There exists $\lambda_0 > 0$ such that for every $\lambda \geq \lambda_0$ and under assumptions (H_1), (H_2) and (H_3) the bilinear form B is coercive. In particular the problem ($\mathscr{E}'$) has a unique solution $u \in H^1(\Gamma)$ for every $f \in H^{-1}(\Gamma)$. Moreover u satisfies $\|u\|_{H^1(\Gamma)} \leq C\|f\|_{H^{-1}}$ for some positive constant C.*

Proof. Our proof use follows the one of [ADLT19, Lemma 2.3] with a slightly different functional framework. Let $u \in H^1(\Gamma; \psi)$. We have

$$\begin{aligned} B(u,u) &= \int_\Gamma a\partial u \partial(u\psi) + \tilde{b}\partial u(u\psi) + c|u|^2\psi \, dx \\ &= \int_\Gamma a|\partial u|^2\psi + c|u|^2\psi + (u\partial u)(a\partial\psi + \tilde{b}\psi) \, dx \\ &\geq \omega\|\partial u\|^2_{L^2(\Gamma;\psi)} + \lambda\|u\|^2_{L^2(\Gamma;\psi)} - (\|a\partial\psi\|_{L^\infty(\Gamma)} + \|\tilde{b}\|_{L^\infty(\Gamma;\psi)}) \int_\Gamma |u\partial u| \, dx. \end{aligned}$$

Let $\epsilon > 0$, using Young's inequality we have

$$\begin{aligned} \int_\Gamma |u\partial u| &\leq \frac{1}{2\epsilon}\|u\|^2_{L^2(\Gamma)} + \frac{\epsilon}{2}\|\partial u\|^2_{L^2(\Gamma)} \\ &\leq \frac{C_\psi}{2\epsilon}\|u\|^2_{L^2(\Gamma;\psi)} + \frac{\epsilon C_\psi}{2}\|\partial u\|^2_{L^2(\Gamma;\psi)}. \end{aligned}$$

Thus, denoting $K = \|a\partial\psi\|_{L^\infty(\Gamma)} + \|\tilde{b}\|_{L^\infty(\Gamma;\psi)}$, we have obtained

$$B(u,u) \geq \left(\lambda - \frac{KC_\psi}{2\epsilon}\right)\|u\|^2_{L^2(\Gamma;\psi)} + \left(\omega - \frac{\epsilon KC_\psi}{2}\right)\|\partial u\|^2_{L^2(\Gamma;\psi)}.$$

Now we can choose ϵ small enough in order to have

$$\left(\omega - \frac{\epsilon KC_\psi}{2}\right) > 0$$

and then λ large enough to also have

$$\left(\lambda - \frac{KC_\psi}{2\epsilon}\right) > 0.$$

We have obtained $B(u,u) \geq C\|u\|^2_{H^1(\Gamma;\psi)}$. We can now apply the Lax-Milgram theorem and there exists a unique solution $u \in H^1(\Gamma;\psi)$, with also $u \in H^1(\Gamma)$ because of the equivalence of the norms, to ($\mathscr{E}'$). Moreover we have

$$C\|u\|^2_{H^1(\Gamma;\psi)} \leq B(u,u) \leq C'\|f\|_{H^{-1}}\|u\|_{H^1(\Gamma;\psi)},$$

and hence

$$\|u\|_{H^1(\Gamma)} \leq \frac{C' C_\psi}{C}\|f\|_{H^{-1}}.$$

□

Remark 3.2.4. *The above result remains true if we only assume $a, \tilde{b} \in L^\infty(\Gamma)$ in the case where we only consider differential operators in divergence form.*

Let us now recall some facts about Hölder continuous functions on a bounded interval.

Lemma 3.2.5. *Let I be a bounded interval of $\mathbb{R}$, $0 < \theta, \gamma \leq 1$ be real numbers and $u \in \mathscr{C}^{0,\theta}(I)$ and $v \in \mathscr{C}^{0,\gamma}(I)$ be functions. Then*

1. *the function $u + v$ belongs to $\mathscr{C}^{0,\eta}(I)$ where $\eta = \min\{\theta, \gamma\}$;*
2. *the function uv belongs to $\mathscr{C}^{0,\eta}(I)$ where $\eta = \min\{\theta, \gamma\}$;*
3. *if $u \geq k > 0$ for some constant k then $\frac{1}{u} \in \mathscr{C}^{0,\theta}(I)$.*

Proof. See Proof A.3.2. □

Proposition 3.2.6 (Regularity of the solution)**.** *In addition to (H_1) and (H_2) assume $f \in L^2(\Gamma)$ and $u \in H^1(\Gamma)$ is a weak solution of ($\mathscr{E}'$) with $\|u\|_{H^1(\Gamma)} \leq K\|f\|_{L^2(\Gamma)}$. Then $u \in H^2(\Gamma)$ and there exists a constant C such that $\|u\|_{H^2(\Gamma)} \leq C\|f\|_{L^2(\Gamma)}$.*

Moreover if $b, c, f \in PC(\Gamma)$ then u belongs to $\mathscr{C}^2(\Gamma)$ and is a classical solution of ($\mathscr{E}_K$) with

$$\|u\|_{\mathscr{C}^2(\Gamma)} \leq C\|f\|_{L^\infty(\Gamma)}.$$

Finally if $b_\alpha, c_\alpha, f_\alpha \in \mathscr{C}^{0,\theta}([0,\ell_\alpha])$, for every $\alpha \in \mathcal{A}$ where $0 < \theta \leq 1$ are real numbers. Then $u \in \mathscr{C}^{2,\theta}(\Gamma)$ and

$$\|u\|_{\mathscr{C}^{2,\theta}(\Gamma)} \leq C \max_{\alpha \in \mathcal{A}} \|f_\alpha\|_{\mathscr{C}^{0,\theta}([0,\ell_\alpha])}.$$

Proof. We already know that $u \in \mathscr{C}(\Gamma)$ and

$$\partial^2 u = \frac{b\partial u + cu - f}{\sigma^2} \tag{3.5}$$

in the sense of distributions inside each $\Gamma_\alpha \setminus \mathcal{V}$. Therefore $\partial^2 u \in L^2(\Gamma)$. This means that $u \in H^2(\Gamma)$.

Assume now that $b, c, f \in PC(\Gamma)$. We already know that $u \in H^2(\Gamma)$ which implies that $\partial u \in PC(\Gamma)$ by standard Sobolev injections (see [Bre11, Theorem 8.8]) and there exists constants C'_α such that $\|u_\alpha\|_{\mathscr{C}^1([0,\ell_\alpha])} \leq C'_\alpha\|f\|_{L^2(\Gamma)}$. Using (3.5) one more time with the additional assumptions on the regularity of the coefficients we get $\partial^2 u \in PC(\Gamma)$ which proves that $u \in \mathscr{C}^2(\Gamma)$ and there exists a constant C' such that

$$\|u\|_{\mathscr{C}^2(\Gamma)} \leq C' \left(\|f\|_{L^2(\Gamma)} + \|f\|_{L^\infty(\Gamma)}\right) \leq \tilde{C}' \|f\|_{L^\infty(\Gamma)}.$$

Finally in the case where $b_\alpha, c_\alpha, f_\alpha \in \mathscr{C}^{0,\theta}([0,\ell_\alpha])$ for every $\alpha \in \mathcal{A}$ the previous case holds and hence $u \in \mathscr{C}^2(\Gamma)$ with $\|u\|_{\mathscr{C}^2(\Gamma)} \leq \|f\|_{L^\infty(\Gamma)}$. In particular we have $u_\alpha \in \mathscr{C}^{1,1}([0,\ell_\alpha])$ and the same argument in conjunction with Lemma 3.2.5 leads to $\partial^2 u_\alpha \in \mathscr{C}^{0,\theta}([0,\ell_\alpha])$. One can see that $\|F_\alpha\|_{\mathscr{C}^{0,\theta}([0,\ell_\alpha])} \leq C'_\alpha \|f_\alpha\|_{\mathscr{C}^{0,\theta}([0,\ell_\alpha])}$ for every $\alpha \in \mathcal{A}$ which then implies that

$$\|u\|_{\mathscr{C}^{0,\theta}(\Gamma)} \leq \tilde{C} \max_{\alpha \in \mathcal{A}} \|f_\alpha\|_{\mathscr{C}^{0,\theta}([0,\ell_\alpha])}.$$

□

Remark 3.2.7. *The proof of these regularity estimates is made much simpler by the fact that the problem is fundamentally one dimensional.*

Lemma 3.2.8. *In addition to (H_1), (H_2) assume $b, c \in PC(\Gamma)$, $c \geq 0$ and $c(x_0) > 0$ for some $x_0 \in \Gamma \setminus \mathcal{V}$. Then the function $u = 0$ is the only (weak) solution to the homogeneous problem*

$$(\mathscr{E}_{Kh}) \qquad \begin{cases} -\sigma^2(x)\partial^2 u(x) + b(x)\partial u(x) + c(x)u(x) = 0 & x \in \Gamma_\alpha \setminus \mathcal{V}, \\ \sum_{\alpha \in \mathcal{A}_i} \gamma_{i,\alpha}\sigma_{|\Gamma_\alpha}(v_i)\partial_\alpha u(v_i) = 0 & \forall i \in \mathcal{I}, \\ u_{|\Gamma_\alpha}(v_i) = u_{|\Gamma_\beta}(v_i) & \forall \alpha, \beta \in \mathcal{A}_i,\ i \in \mathcal{I}. \end{cases}$$

Furthermore if $c = 0$, the only possible solutions are constants.

Proof. By the same argument as for Proposition 3.2.6 we have that any weak solution of the inhomogeneous problem is in fact a classical solution. Then the result is a direct consequence of Theorem 3.1.3. □

We now state the general existence and uniqueness theorem for weak solutions of the elliptic problem. This is a special case the result for ramified spaces proved in [Nic88, Theorem 2.2].

Theorem 3.2.9. *In addition to (H_1), (H_2) assume $b, c \in PC(\Gamma)$, $c \geq 0$ and $c(x_0) > 0$ for some $x_0 \in \Gamma \setminus \mathcal{V}$. Then for any $f \in H^{-1}(\Gamma)$, there exists a unique $u \in H^1(\Gamma)$ solution of ($\mathscr{E}'$). Moreover there exists a constant $C > 0$ such that $\|u\|_{H^1(\Gamma)} \leq C\|f\|_{H^{-1}}$.*

Proof. The proof is reminiscent of [ADLT19, Lemma 2.3] and [Eva10, Theorem 4 p.323] and is presented in Proof A.3.3. □

Proposition 3.2.10. *In addition to (H_1), (H_2) assume $b, c, f \in PC(\Gamma)$, $c \geq 0$ and $c > 0$ in a neighborhood of v_i for every $i \in \mathcal{I}$. Then if f is nonnegative on Γ then the solution u of ($\mathscr{E}_K$) is also non-negative on Γ.*

Proof. The proof is inspired by an argument made in [ADLT19, Lemma 3.3] and postponed to Proof A.3.4. □

3.3 A dual equation

The purpose of this section is to study the following problem

$$\begin{cases} -\partial\left(a\partial w\right)+\partial\left(bw\right)=0, & \text{on } \Gamma\setminus\mathcal{V}, \\ \frac{w_{|\Gamma_\alpha}(v_i)}{\gamma_{i,\alpha}}=\frac{w_{|\Gamma_\beta}(v_i)}{\gamma_{i,\beta}}, & \forall\alpha,\beta\in\mathcal{A}_i,\ \forall i\in\mathcal{I}, \\ \sum_{\alpha\in\mathcal{A}_i} a_{|\Gamma_\alpha}\partial_\alpha w_{|\Gamma_\alpha}(v_i)-n_{i,\alpha}w_{|\Gamma_\alpha}(v_i)a_{|\Gamma_\alpha}(v_i)=0 & \forall i\in\mathcal{I}, \end{cases} \tag{3.6}$$

where we assume $a, b \in PC(\Gamma)$ with $a \geq \omega > 0$, $\gamma_{i,\alpha} \in (0, +\infty)$ for every $\alpha \in \mathcal{A}_i$ and $i \in \mathcal{I}$. The motivation for the study of this problem is the stationary Fokker-Plank-Kolmogorov equation which will be derived in Chapter 5.

For this, following [ADLT19], we will use the following function space

$$W=\left\{w\in H_b^1(\Gamma):\ \frac{w_{|\Gamma_\alpha}(v_i)}{\gamma_{i,\alpha}}=\frac{w_{|\Gamma_\beta}(v_i)}{\gamma_{i,\beta}},\ \forall\alpha,\beta\in\mathcal{A}_i,\ \forall i\in\mathcal{I}\right\}.$$

It becomes an Hilbert space when provided with the inner product

$$(w,v)_W=(w,v)_{L^2(\Gamma)}+(\partial w,\partial v)_{L^2(\Gamma)}.$$

In order to derive the weak formulation of (3.6) we multiply the equation by $v \in H^1(\Gamma)$ and integrate over Γ, assuming that $w \in W$ with $w_\alpha \in \mathscr{C}^1([0, \ell_\alpha])$ for every $\alpha \in \mathcal{A}$ is a solution of the problem

$$\begin{aligned} 0 &= \int_\Gamma -\partial\left(a\partial w\right)v+\partial\left(bw\right)v\,dx \\ &= \sum_{\alpha\in\mathcal{A}}\left[-a_\alpha\partial w_\alpha v_\alpha+b_\alpha w_\alpha v_\alpha\right]_0^{\ell_\alpha}+\int_\Gamma a\partial w\partial v-bw\partial v\,dx \\ &= \sum_{i\in\mathcal{I}} v(v_i)\left[\sum_{\alpha\in\mathcal{A}_i} n_{i,\alpha}b_{|\Gamma_\alpha}(v_i)w_{|\Gamma_\alpha}(v_i)-a_{|\Gamma_\alpha}(v_i)\partial_\alpha w(v_i)\right]+\int_\Gamma a\partial w\partial v-bw\partial v\,dx \\ &= \int_\Gamma a\partial w\partial v-bw\partial v\,dx. \end{aligned}$$

Now consider the function $\phi \in PC(\Gamma)$ satisfying

$$\begin{cases} \phi_\alpha \text{ is affine on } [0,\ell_\alpha] \text{ for every } \alpha\in\mathcal{A}, \\ \phi_{|\Gamma_\alpha}(v_i)=\frac{1}{\gamma_{i,\alpha}} \quad \forall\alpha\in\mathcal{A}_i,\forall i\in\mathcal{I}, \end{cases} \tag{3.7}$$

and notice that for every $w \in W$ the function $v = w\phi$ belongs to $H^1(\Gamma)$. Then if we write

$$L^\star w=-\partial(a\partial w)+\partial(bw)$$

the computation above can be summarized by

$$(L^\star w, v)_{L^2(\Gamma;\phi)}=B(w,v)=0 \tag{3.8}$$

for every $v \in W$ where B is the continuous bilinear form defined on $W \times W$ by

$$B(w,v) = \int_\Gamma a\partial w \partial(v\phi) - bw\partial(v\phi). \tag{3.9}$$

Conversely if $w \in W$ is such that (3.8) holds for every $v \in W$ one can prove that if w is smooth enough, choosing the right test functions, then if is a solution of (3.6).

We define W_ϕ to be the function space W provided with the equivalent inner product

$$(w,v)_{W_\phi} = (w,v)_{L^2(\Gamma;\phi)} + (\partial w, \partial v)_{L^2(\Gamma;\phi)}.$$

Proposition 3.3.1. *There exists $\lambda_0 > 0$ such that for every $\lambda \geq \lambda_0$ the bilinear form*

$$W_\phi \times W_\phi \ni (w,v) \mapsto B_\lambda(w,v) = B(w,v) + \lambda(w,v)_{L^2(\Gamma;\phi)}$$

is coercive and continuous. In particular there exists a unique weak solution $w \in W$ to the problem

$$\begin{cases} -\partial\left(a\partial w\right) + \partial\left(bw\right) + \lambda w = h, & \text{on } \Gamma \setminus \mathcal{V}, \\ \frac{w_{|\Gamma_\alpha}(v_i)}{\gamma_{i,\alpha}} = \frac{w_{|\Gamma_\beta}(v_i)}{\gamma_{i,\beta}}, & \forall \alpha, \beta \in \mathcal{A}_i, \ \forall i \in \mathcal{I}, \\ \sum_{\alpha \in \mathcal{A}_i} a_{|\Gamma_\alpha} \partial_\alpha w_{|\Gamma_\alpha}(v_i) - n_{i,\alpha} w_{|\Gamma_\alpha}(v_i) a_{|\Gamma_\alpha}(v_i) = 0 & \forall i \in \mathcal{I}, \end{cases} \tag{3.10}$$

for every $\lambda \geq \lambda_0$ and $h \in L^2(\Gamma)$. Furthermore the following estimate holds

$$\|w\|_W \leq C\|h\|_{L^2(\Gamma)}.$$

Proof. The proof follows the same lines as the one for Theorem 3.2.3 (see also [ADLT19, Theorem 2.7]) and can be found in Proof A.3.5. □

We now come to the main result of this section.

Theorem 3.3.2. *There exists a unique weak solution $w \in W$ to (3.6) such that w is nonnegative and*

$$\int_\Gamma w \, dx = 1.$$

Proof. Our proof is mostly identical to the one of [ADLT19, Theorem 2.7] and we hence postpone it to Proof A.3.6. □

From the usual Sobolev inequalities we know that the solution w of (3.6) belongs to $PC(\Gamma)$. Moreover the fact that w satisfies

$$-\partial\left(a\partial w\right) + \partial\left(bw\right) = 0, \quad \text{on } \Gamma \setminus \mathcal{V},$$

in the sense of distributions implies that

$$-a_\alpha \partial w_\alpha + b_\alpha w_\alpha = c_\alpha$$

for some constant $c_\alpha \in \mathbb{R}$ for every $\alpha \in \mathcal{A}$ (see [Hö3, Theorem 3.1.4]). Therefore we have $w_\alpha \in \mathscr{C}^1([0,\ell_\alpha])$ for every $\alpha \in \mathcal{A}$. It will we convenient to introduce the following function space

$$\mathcal{W} = \left\{ w \in W : \; w_\alpha \in \mathscr{C}^1([0,\ell_\alpha]) \text{ for every } \alpha \in \mathcal{A} \right\}.$$

In particular the solution w satisfies the transmission condition in the classical sense. We thus have the following regularity result.

Proposition 3.3.3. *The weak solution w of (3.6) obtained in Theorem 3.3.2 belongs to $\mathcal{W}$. Furthermore if $b \in W_b^{1,\infty}(\Gamma)$ then the solution belongs to $\mathcal{W} \cap H_b^2(\Gamma)$.*

Chapter 4

Second order linear parabolic equations

Our main goal in this chapter is to study the well-posedness of linear parabolic equations on networks with Kirchhoff transmission conditions. That is problems of the form

$$(4.1)\qquad \begin{cases} \partial_t u - \sigma^2 \partial_x^2 u + b\partial_x u + cu = f & \forall t \in (0,T),\ x \in \Gamma \setminus \mathcal{V}, \\ \sum_{\alpha \in \mathcal{A}_i} \gamma_{i,\alpha} \sigma^2_{|\Gamma_\alpha}(\nu_i) \partial_\alpha u(t,\nu_i) = 0 & \forall i \in \mathcal{I},\ \forall t \in (0,T), \\ u_{|\Gamma_\alpha}(t,\nu_i) = u_{|\Gamma_\beta}(t,\nu_i) & \forall \alpha,\beta \in \mathcal{A}_i, \forall i \in \mathcal{I},\ \forall t \in (0,T), \\ u(0,\cdot) = u_0. \end{cases}$$

To this end we apply the theory of semigroups of bounded linear operators (we recall the necessary facts about semigroups in Appendix B) to prove the well-posedness of the problem in $L^2(\Gamma)$ in the case where the coefficients σ, b and c are independent of time. In this we are strongly inspired by [Paz83, Section 7.2] and [Eva10, Section 7.1]. Note that similar results on ramified spaces were proved in [vBN96]. See also [Mug14] for semigroups on networks. Weak solutions for linear parabolic problems were obtained in [ADLT20] using the Galerkin method. We show how our approach also allows to build weak solutions of the problem. Finally see [vB88] for another approach to classical solutions of linear parabolic equations on networks.

We end this introduction with a remark which is often useful in the context of evolution equations. Let $u : I \times \Gamma \to \mathbb{R}$ be a function where I is an interval of $\mathbb{R}_+$ and assume that for every $t \in I$ then function $u(t,\cdot)$ belong to a function space X, which we may assume is a Banach space. We can then identify the function u with

$$\boldsymbol{u} : I \ni t \mapsto \boldsymbol{u}(t) = u(t,\cdot) \in X.$$

With this point of view we may interpret (4.1), if we forget the Kirchhoff condition for now, as a Cauchy problem in the Banach space X

$$(4.2)\qquad \begin{cases} \boldsymbol{u}'(t) + L\boldsymbol{u}(t) = \boldsymbol{f}(t), \\ \boldsymbol{u}(0) = u_0. \end{cases}$$

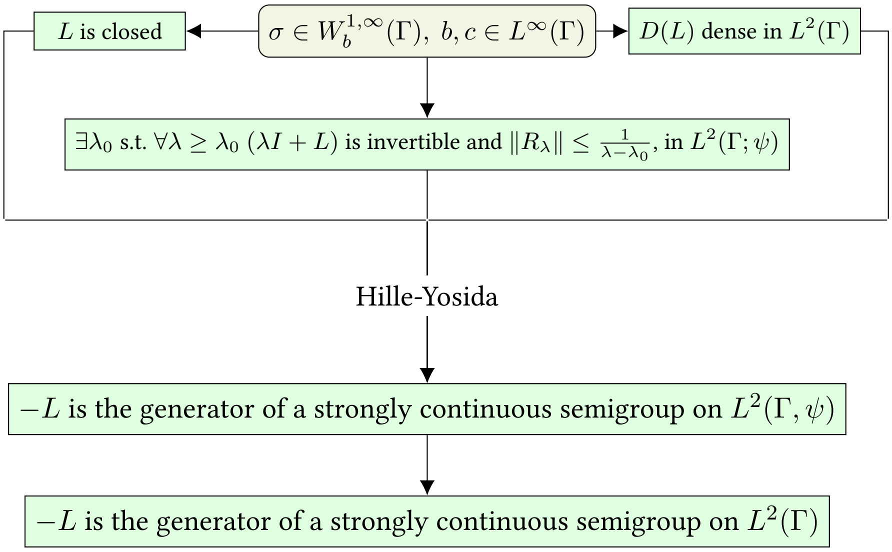

Figure 4.1 – Main steps of the proof of Theorem 4.1.4.

where L is a linear operator on X with domain $D(L)$. In what follows we will write $\partial_t u(t, \cdot)$ to mean $\boldsymbol{u}'(t)$ and $Lu(t, \cdot)$ for $L\boldsymbol{u}(t)$. This identification leads us to considers functions spaces for functions $\boldsymbol{u} : I \to X$ such as $L^2((0, T), H^1(\Gamma))$. The reader can refer to [Eva10, Section 5.9.2] or [LM72, Chapter 1] for precise definitions and properties of these function spaces.

4.1 Existence of a strongly continuous semigroup and well-posedness for regular initial data

Let $L = -\sigma^2 \partial_x^2 + b\partial_x + c$ be an elliptic differential operator on Γ (see Chapter 3). Our goal here is to study parabolic equations of the form

$$(\mathscr{P}_K) \qquad \begin{cases} \partial_t u(t, x) + Lu(t, x) = f(t, x) & \text{for } t \in (0, T),\ x \in \Gamma \setminus \mathcal{V}, \\ \sum_{\alpha \in \mathcal{A}_i} \gamma_{i,\alpha} \sigma^2_{|\Gamma_\alpha}(v_i) \partial_\alpha u(t, v_i) = 0 & \forall i \in \mathcal{I} \text{ for every } t \in (0, T), \\ u_{|\Gamma_\alpha}(t, v_i) = u_{|\Gamma_\beta}(t, v_i) & \forall \alpha, \beta \in \mathcal{A}_i, \forall i \in \mathcal{I} \text{ for every } t \in (0, T), \\ u(0, \cdot) = u_0. \end{cases}$$

We proceed by first showing that $-L$ is the infinitesimal generator of a strongly continuous semiroup on $L^2(\Gamma)$. The scheme of the proof is very close to the proof in the case of uniformly elliptic operators on smooth domains in $\mathbb{R}^d$ and is presented in Fig. 4.1. In this chapter we suppose that σ, b and c are independent of the time variable. We also assume $\sigma_{|\Gamma_\alpha} \in W^{1,\infty}(\Gamma_\alpha)$ for every $\alpha \in \mathcal{A}$ and is far from zero, *i.e.* $\sigma^2 \geq \omega > 0$ for some positive constant ω. We also suppose $b, c \in L^\infty(\Gamma)$ and $f, u_0 \in L^2(\Gamma)$.

Recall from Section 3.2 that to the elliptic differential operator L, which can be written in divergence form

$$Lu = -\partial(a\partial u) + \tilde{b}\partial u + cu,$$

with Kirchhoff transmission condition we can associate a bilinear form B defined on $H^1(\Gamma) \times H^1(\Gamma)$ by

$$B(u,v) = \int_\Gamma a(x)\partial u \partial(v\psi) + \tilde{b}\partial u v\psi + cuv\psi \, dx. \tag{4.3}$$

In what follows we consider the following domain for L :

$$D(L) = \left\{ u \in H^2(\Gamma) : \sum_{\alpha \in \mathcal{A}_i} \gamma_{i,\alpha}\sigma^2_{|\Gamma_\alpha}(v_i)\partial_\alpha u(v_i) = 0 \; \forall i \in \mathcal{I} \right\}.$$

Recall also that we consider the weighted space $L^2(\Gamma;\psi)$ equipped with the following inner product

$$(u,v)_{L^2(\Gamma;\psi)} = \int_\Gamma u(x)v(x)\psi(x) \, dx$$

which is equivalent to the usual inner product on $L^2(\Gamma)$ and we have for every $u, v \in D(L)$

$$B(u,v) = (u,v)_{L^2(\Gamma;\psi)}.$$

Definition 4.1.1. A *semigroup solution* of ($\mathscr{P}_K$) is a function

$$u \in \mathscr{C}^1((0,T), L^2(\Gamma)) \cap \mathscr{C}((0,T), D(L)) \cap \mathscr{C}([0,T], L^2(\Gamma))$$

satisfying

$$\begin{cases} \partial_t u(t,\cdot) + Lu(t,\cdot) = f(t,\cdot) \quad \forall t \in (0,T), \\ u(0,\cdot) = u_0, \end{cases}$$

in the sense of differential equations in $L^2(\Gamma)$. ◁

Lemma 4.1.2. *The domain $D(L)$ is dense in $L^2(\Gamma;\psi)$.*

Proof. See Proof A.4.1. □

Lemma 4.1.3. *The differential operator L is closed in $L^2(\Gamma;\psi)$.*

Proof. See Proof A.4.2. □

We come to the main result of this section.

Theorem 4.1.4. *There exists a positive constant λ_0 such that $-L$ is the infinitesimal generator of a strongly continuous semigroup $(T_t)_{t\geq 0}$ on $L^2(\Gamma)$ satisfying*

$$\|T_t\|_{\mathscr{L}(L^2(\Gamma))} \leq Ce^{\lambda_0 t}.$$

Proof. We follow the argument from [Eva10, Theorem 5 p.444].From Theorem 3.2.3 we know that there exists a positive constant λ_0 such that the bilinear form

$$\hat{B}(u,v) = B(u,v) + \lambda_0 (u,v)_{L^2(\Gamma;\psi)}$$

is coercive on $H^1(\Gamma) \times H^1(\Gamma)$. In particular this implies that

$$-\lambda_0 \|u\|^2_{L^2(\Gamma;\psi)} \leq C\|u\|^2_{H^1(\Gamma)} - \lambda_0 \|u\|^2_{L^2(\Gamma;\psi)} \leq B(u,u). \tag{4.4}$$

In this case Theorem 3.2.3 and Proposition 3.2.6 imply that for every $\lambda > \lambda_0$ and $f \in L^2(\Gamma)$, there exists a unique $u \in D(L)$ such that

$$\hat{B}(u,v) + \lambda (u,v)_{L^2(\Gamma;\psi)} = (f,v)_{L^2(\Gamma;\psi)} \tag{4.5}$$

for every $v \in H^1(\Gamma;\psi)$. Therefore, combining (4.4) and (4.5), we have

$$(\lambda - \lambda_0)\|u\|_{L^2(\Gamma;\psi)} \leq \|f\|_{L^2(\Gamma;\psi)}.$$

Hence the resolvent set of $-L$ satisfies $(\lambda_0, \infty) \subset \rho(-L)$ and the resolvent is such that

$$\|R_\lambda\|_{\mathscr{L}(L^2(\Gamma;\psi))} \leq \frac{1}{\lambda - \lambda_0}$$

for every $\lambda > \lambda_0$. This, along with Lemmas 4.1.2 and 4.1.3, allows us to deduce form the Hille-Yosida theorem (more precisely from Corollary B.1.9) that $-L$ is the generator of a strongly continuous semigroup $(T_t)_{t\geq 0}$ on $L^2(\Gamma;\psi)$ and

$$\|T_t\|_{\mathscr{L}(L^2(\Gamma;\psi))} \leq e^{\lambda_0 t}.$$

Then $(T_t)_{t\geq 0}$ is also a strongly continuous semigroup on $L^2(\Gamma)$ and satisfies

$$\|T_t\|_{\mathscr{L}(L^2(\Gamma))} \leq Ce^{\lambda_0 t}.$$

□

Corollary 4.1.5. *Assume $f \in \mathscr{C}^1([0,T], L^2(\Gamma))$. Then for every $u_0 \in D(L)$ the problem*

$$\begin{cases} \partial_t u(t,x) + Lu(t,x) = f(t,x) & \text{for } t \in (0,T),\ x \in \Gamma \setminus \mathcal{V}, \\ \sum_{\alpha \in \mathcal{A}_i} \gamma_{i,\alpha} \sigma^2_{|\Gamma_\alpha}(v_i) \partial_\alpha u(t,v_i) = 0 & \forall i \in \mathcal{I} \text{ for every } t \in (0,T), \\ u_{|\Gamma_\alpha}(t,v_i) = u_{|\Gamma_\beta}(t,v_i) & \forall \alpha,\beta \in \mathcal{A}_i, \forall i \in \mathcal{I} \text{ for every } t \in (0,T), \\ u(0,\cdot) = u_0(\cdot). \end{cases} \tag{4.6}$$

has a unique semigroup solution

$$u \in \mathscr{C}^1((0,T), L^2(\Gamma)) \cap \mathscr{C}([0,T), H^2(\Gamma)).$$

Proof. Apply Theorem B.3.3 with the semigroup from Theorem 4.1.4. □

We now want to obtain better estimates than the simple L^2 ones which follow from the operator norm of the semigroup. Recall that the solution is given by Duhamel's formula

$$u(t,\cdot) = T_t u_0 + \int_0^t T_{t-s} f(s,\cdot)\, ds$$

where $(T_t)_{t\geq 0}$ is the strongly continuous semigroup obtained in Theorem 4.1.4. This implies that

$$\begin{aligned}
\|u(t,\cdot)\|^2_{L^2(\Gamma;\psi)} &\leq \|T_t u_0\|^2_{L^2(\Gamma;\psi)} + \int_0^t \|T_{t-s} f(s,\cdot)\|^2_{L^2(\Gamma;\psi)}\, ds \\
&\leq e^{2\lambda_0 t}\|u_0\|^2_{L^2(\Gamma;\psi)} + \int_0^t e^{2\lambda_0(t-s)}\|f(s,\cdot)\|^2_{L^2(\Gamma;\psi)}\, ds \\
&\leq e^{2\lambda_0 T}\left(\|u_0\|^2_{L^2(\Gamma;\psi)} + \int_0^T \|f(s,\cdot)\|^2_{L^2(\Gamma;\psi)}\, ds\right) \\
&= e^{2\lambda_0 T}\left(\|u_0\|^2_{L^2(\Gamma;\psi)} + \|f\|^2_{L^2(0,T,L^2(\Gamma;\psi))}\right)
\end{aligned}$$

and therefore

$$\|u\|_{\mathscr{C}([0,T],L^2(\Gamma;\psi))} \leq C\left(\|u_0\|_{L^2(\Gamma;\psi)} + \|f\|_{L^2(0,T,L^2(\Gamma;\psi))}\right). \tag{4.7}$$

Remark that this also implies that

$$\|u\|_{L^2(0,T,L^2(\Gamma;\psi))} \leq T\|u\|_{\mathscr{C}([0,T],L^2(\Gamma;\psi))} \leq CT\left(\|u_0\|_{L^2(\Gamma;\psi)} + \|f\|_{L^2(0,T,L^2(\Gamma;\psi))}\right). \tag{4.8}$$

We know from Corollary 4.1.5 that $u \in \mathscr{C}^1((0,T), L^2(\Gamma;\psi))$. Hence we may write

$$\begin{aligned}
\frac{d}{dt}\left(\frac{1}{2}\|u(t,\cdot)\|^2_{L^2(\Gamma;\psi)}\right) &= (\partial_t u(t,\cdot), u(t,\cdot))_{L^2(\Gamma;\psi)} \\
&= (-Lu(t,\cdot), u(t,\cdot))_{L^2(\Gamma;\psi)} + (f(t,\cdot), u(t,\cdot))_{L^2(\Gamma,\psi)}
\end{aligned} \tag{4.9}$$

As u satisfied the Kirchhoff condition we can integrate by parts to find

$$-(Lu(t,\cdot), u(t,\cdot))_{L^2(\Gamma;\psi)} = -B(u(t,\cdot), u(t,\cdot))$$

where B is the bilinear form associated with the elliptic operator L with Kirchhoff condition (see (3.4)). According to Theorem 3.2.3 there exists a real constant λ_0 such that $B(\cdot,\cdot) + \lambda_0(\cdot,\cdot)_{L^2(\Gamma;\psi)}$ is coercive. This implies that there exists a positive constant C such that

$$C\|u(t,\cdot)\|^2_{H^1(\Gamma;\psi)} - \lambda_0\|u(t,\cdot)\|^2_{L^2(\Gamma;\psi)} \leq B(u(t,\cdot), u(t,\cdot)).$$

This yields, using also Young's inequality on $(f(t,\cdot), u(t,\cdot))_{L^2(\Gamma,\psi)}$,

$$\frac{d}{dt}\left(\frac{1}{2}\|u(t,\cdot)\|^2_{L^2(\Gamma;\psi)}\right) + C\|u(t,\cdot)\|_{H^1(\Gamma;\psi)} \leq (\lambda_0+1)\|u(t,\cdot)\|^2_{L^2(\Gamma;\psi)} + \frac{1}{4}\|f(t,\cdot)\|_{L^2(\Gamma;\psi)}$$

and after integration

$$
\begin{aligned}
(4.10)\qquad & \frac{1}{2}\|u(t,\cdot)\|^2_{L^2(\Gamma;\psi)} + C\int_0^t \|u(s,\cdot)\|^2_{H^1(\Gamma;\psi)}\,ds \\
& \leq \frac{1}{2}\|u_0\|^2_{L^2(\Gamma;\psi)} + (\lambda_0+1)\int_0^t \|u(s,\cdot)\|^2_{L^2(\Gamma;\psi)}\,ds + \frac{1}{4}\int_0^t \|f(s,\cdot)\|^2_{L^2(\Gamma;\psi)}\,ds,
\end{aligned}
$$

where we used the fact that $u \in \mathscr{C}([0,T), L^2(\Gamma;\psi))$ to obtain that $\|u(0,\cdot)\|_{L^2(\Gamma;\psi)} = \|u_0\|_{L^2(\Gamma;\psi)}$. In particular we have that

$$
\int_0^T \|u(s,\cdot)\|^2_{H^1(\Gamma;\psi)}\,ds \leq \frac{1}{2C}\|u_0\|^2_{L^2(\Gamma;\psi)} + \frac{1}{4C}\|f\|^2_{L^2(0,T,L^2(\Gamma;\psi))} + \frac{\lambda_0+1}{C}\|u\|^2_{L^2(0,T,L^2(\Gamma;\psi))}.
$$

Using (4.8) we conclude that

$$
(4.11)\qquad \|u\|_{L^2(0,T,H^1(\Gamma;\psi))} \leq C\left(\|u_0\|_{L^2(\Gamma;\psi)} + \|f\|_{L^2(0,T,L^2(\Gamma;\psi))}\right).
$$

The obtain estimates on $\partial_t u$ notice that multiplying by $v \in H^1(\Gamma;\psi)$ in the equation satisfied by u and integrating with respect to x we find, using integration by parts as we did for (4.9)

$$
\begin{aligned}
(4.12)\qquad (\partial_t u(t,\cdot), v)_{L^2(\Gamma;\psi)} &= -B(u(t,\cdot),v) + (f(t,\cdot),v)_{L^2(\Gamma;\psi)} \\
&\leq k\|u(t,\cdot)\|_{H^1(\Gamma;\psi))}\|v\|_{H^1(\Gamma;\psi)} + \|f(t,\cdot)\|_{L^2(\Gamma;\psi)}\|v\|_{H^1(\Gamma;\psi)}
\end{aligned}
$$

for every $0 < t < T$. As it is true for every $v \in H^1(\Gamma;\psi)$, this implies

$$
\sup_{v\in H^1(\Gamma;\psi)} \frac{(\partial_t u(t,\cdot), v)_{L^2(\Gamma;\psi)}}{\|v\|_{H^1(\Gamma;\psi)}} \leq k\|u(t,\cdot)\|_{H^1((\Gamma;\psi)} + \|f(t,\cdot)\|_{L^2(\Gamma;\psi)}.
$$

Therefore we have

$$
(4.13)\qquad \|\partial_t u(t,\cdot)\|_{H^{-1}(\Gamma;\psi)} \leq k\|u(t,\cdot)\|_{H^1((\Gamma;\psi)} + \|f(t,\cdot)\|_{L^2(\Gamma;\psi)}
$$

for every $0 < t \leq T$ and after integration

$$
\begin{aligned}
(4.14)\qquad \|\partial_t u\|_{L^2(0,T,H^{-1}(\Gamma;\psi))} &\leq C\left(\|u\|_{L^2(0,T,H^1(\Gamma;\psi))} + \|f\|_{L^2((0,T),L^2(\Gamma;\psi))}\right) \\
&\leq C\left(\|u_0\|_{L^2(\Gamma;\psi)} + \|f\|_{L^2(0,T,L^2(\Gamma;\psi))}\right).
\end{aligned}
$$

Using the equivalence of the norms in $L^2(\Gamma;\psi)$ and $L^2(\Gamma)$ we have proved the following theorem.

Theorem 4.1.6. *Let $u_0 \in D(L)$, $f \in \mathscr{C}^1([0,T], L^2(\Gamma))$ and u be the solution from Corollary 4.1.5. Then u satisfies*

$$
\|u\|_{\mathscr{C}([0,T],L^2(\Gamma))} + \|u\|_{L^2(0,T,H^1(\Gamma))} + \|\partial_t u\|_{L^2(0,T,H^1(\Gamma))} \leq C\left(\|u_0\|_{L^2(\Gamma)} + \|f\|_{L^2(0,T,L^2(\Gamma))}\right).
$$

4.2 From strongly continuous semigroups to weak solutions

We want to extend our the previous results to more general data $u_0 \in L^2(\Gamma)$ and $f \in L^2((0,T), L^2(\Gamma))$. For this we introduce a weaker notion of solution. Existence of such weak solutions was obtained in [ADLT20] by the use of the Galerkin method.

Definition 4.2.1. A *weak solution* of ($\mathscr{P}_K$) is a function

$$u \in L^2((0,T), H^1(\Gamma)), \text{ with } \partial_t u \in L^2((0,T), H^{-1}(\Gamma))$$

such that

$$\langle \partial_t u(t,\cdot), v\rangle_{H^{-1},H^1} + B(u(t,\cdot), v) = (f, v)_{L^2(\Gamma;\psi)}$$

for a.e. $t \in (0,T)$ and $u(0,\cdot) = u_0$ in $L^2(\Gamma)$, where B is the bilinear form associated to the elliptic differential operator L. ◁

Theorem 4.2.2. *Let $u_0 \in L^2(\Gamma)$ and $f \in L^2((0,T), L^2(\Gamma))$. Then there exists a weak solution*

$$u \in L^2((0,T), H^1(\Gamma)), \quad \partial_t u \in L^2(0,T, H^{-1}(\Gamma))$$

of ($\mathscr{P}_K$) with

$$\|u\|_{L^2(0,T,H^1(\Gamma))} + \|\partial_t u\|_{L^2(0,T,H^{-1}(\Gamma))} \leq C\left(\|u_0\|_{L^2(\Gamma)} + \|f\|_{L^2(0,T,L^2(\Gamma))}\right).$$

Proof. See Proof A.4.3 □

Theorem 4.2.3. *Let $u_0 \in L^2(\Gamma)$ and $f \in L^2((0,T), L^2(\Gamma))$. Then there exists a unique weak solution to ($\mathscr{P}_K$).*

Proof. It is enough to prove that the function $u = 0$ is the only weak solution for $u_0 = f = 0$. We have

$$\begin{aligned}\frac{d}{dt}\left(\frac{1}{2}\|u(t,\cdot)\|^2_{L^2(\Gamma;\psi)}\right) &+ B(u(t,\cdot), u(t,\cdot)) \\ &= \langle \partial_t u(t,\cdot), u(t,\cdot)\rangle_{H^{-1},H^1} + B(u(t,\dot{)}, u(t,\cdot)) = 0.\end{aligned}$$

Since $B(u(t,\cdot), u(t,\cdot)) \geq -\lambda_0\|u(t,\cdot)\|^2_{L^2(\Gamma;\psi)}$ we obtain that

$$\frac{d}{dt}\left(\|u(t,\cdot)\|^2_{L^2(\Gamma;\psi)}\right) \leq 2\lambda_0\|u(t,\cdot)\|^2_{L^2(\Gamma;\psi)}.$$

We conclude using the differential form of Gronwall's lemma. □

4.3 Existence of an analytic semigroup and well-posedness for general initial data

In order to extend the existence of semigroup solutions to initial data in $L^2(\Gamma)$ instead of $D(L)$ we have to prove that the strongly continuous semigroup obtained in Theorem 4.1.4 is in fact analytic. For this purpose we need to consider complex valued functions. Hence the spaces $\hat{L}^2(\Gamma), \hat{L}^2(\Gamma;\psi), \hat{H}^1(\Gamma)$ and $\hat{H}^1(\Gamma;\psi)$ will denote the natural extension to complex valued functions of the functions spaces defined previously. Note that these spaces are vector spaces on $\mathbb{C}$ instead of $\mathbb{R}$. However we still consider functions σ, b, c and f which are real valued.

Our first step is to study the complex elements of the resolvent set $\rho(-L)$. We define the sesquilinear form associated to complex elliptic problem by

$$\hat{B}(u,v) = B(u,\bar{v}) \quad \forall (u,v) \in \hat{H}^1(\Gamma;\psi) \times \hat{H}^1(\Gamma;\psi). \tag{4.15}$$

We now state the main theorem of this section. An analogue theorem for more general ramified spaces was proved in [vBN96, Theorem 3.4].

Theorem 4.3.1. *The operator $-L$ is the infinitesimal generator of an analytic semigroup.*

Proof. The proof is mostly identical to one for semigroups generated by uniformly elliptic operators on domains in $\mathbb{R}^d$ as presented in [Paz83, Theorem 7.2.7] and is therefore sent to Proof A.4.4. □

We deduce an improved existence and uniqueness theorem for linear parabolic equations. For ramified spaces see [vBN96, Corollary 3.5].

Corollary 4.3.2. *Assume $f \in W^{1,\infty}((0,T), L^2(\Gamma))$. Then for every $u_0 \in L^2(\Gamma)$ the problem*

$$\begin{cases} \partial_t u(t,x) + Lu(t,x) = f(t,x) & \text{for } t \in (0,T),\ x \in \Gamma \setminus \mathcal{V}, \\ \sum_{\alpha \in \mathcal{A}_i} \gamma_{i,\alpha} \sigma^2_{|\Gamma_\alpha}(v_i) \partial_\alpha u(t,v_i) = 0 & \forall i \in \mathcal{I} \text{ for every } t \in (0,T), \\ u_{|\Gamma_\alpha}(t,v_i) = u_{|\Gamma_\beta}(t,v_i) & \forall \alpha,\beta \in \mathcal{A}_i, \forall i \in \mathcal{I} \text{ for every } t \in (0,T), \\ u(0,\cdot) = u_0. \end{cases} \tag{4.16}$$

has a unique semigroup solution

$$u \in \mathscr{C}^1((0,T), L^2(\Gamma)) \cap \mathscr{C}((0,T), H^2(\Gamma)) \cap \mathscr{C}([0,T), L^2(\Gamma)).$$

Proof. Apply Theorem B.3.4 with the semigroup from Theorem 4.3.1. □

We can use the same arguments as for Theorem 4.1.6, this time with arbitrary $u_0 \in L^2(\Gamma)$ to obtain

$$\|u\|_{\mathscr{C}([0,T],L^2(\Gamma))} + \|u\|_{L^2(0,T,H^1(\Gamma))} + \|\partial_t u\|_{L^2(0,T,H^{-1}(\Gamma))} \le C\left(\|u_0\|_{L^2(\Gamma)} + \|f\|_{L^2(0,T,L^2(\Gamma))}\right).$$

Chapter 5

Stochastic processes on networks

Usually the stochastic process representing the representative agent in mean field games is assumed to be the solution of a stochastic differential equation. However as we want to consider a network as the state space, it is not clear how one can define such an equation in this case. A solution to this problem is to shift the point of view. Indeed it is standard to see solutions of stochastic differential equations as continuous Markov processes. The workaround is then to use Feller's theory to build a Markov process which we hope to have a behavior similar to the one of the solution of a SDE. The construction of such a process was first announced in [FW93] where the proof is only sketched, and studied in more details in [FS00]. The main steps of the proof are presented in Fig. 5.1 In what follows we use the results on second order elliptic equations on networks obtained in Chapter 3 to prove the existence of the process following [FW93]. This is a standard approach, see [Tai20, Chapter 3] for instance. We then derive some of the properties of the process and derive the Fokker-Planck-Kolmogorov equations satisfied by its transition probabilities and its invariant measure. We will heavily rely on facts and notations about semigroups of bounded linear operators which are recalled in Appendix B.

5.1 Existence of diffusion processes on networks

Consider two functions $\sigma, b \in PC(\Gamma)$ with $\sigma_\alpha \in \mathscr{C}^{0,1}([0,\ell_\alpha])$ for every $\alpha \in \mathcal{A}$. Assume also that there exists a positive constant ω such that $\sigma^2_{|\Gamma_\alpha} \geq \omega$ for every $\alpha \in \mathcal{A}$. Let also $(\gamma_{i,\alpha})_{\alpha\in\mathcal{A}_i}$ be families of positive numbers for each $i \in \mathcal{I}$ such that $\sum_{\alpha\in\mathcal{A}_i} \gamma_{i,\alpha}\sigma^2_{|\Gamma_\alpha}(v_i) = 1$. In what follows we denote $p_{i,\alpha} = \gamma_{i,\alpha}\sigma^2_{|\Gamma_\alpha}(v_i)$.

We define the following linear differential operator

$$Lu(x) = L_\alpha u(x) = \sigma^2_{|\Gamma_\alpha}(x)\partial^2 u(x) + b_{|\Gamma_\alpha}(x)\partial u(x), \quad \text{if } x \in \Gamma_\alpha,$$

with domain

$$D(L) = \left\{ u \in \mathscr{C}^2(\Gamma) : Lu \in \mathscr{C}(\Gamma),\ \sum_{\alpha\in\mathcal{A}_i} p_{i,\alpha}\partial_\alpha u(v_i) = 0 \text{ for all } i \in I \right\}.$$

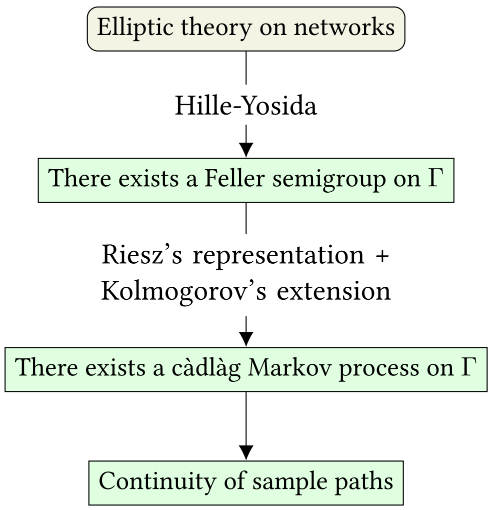

Figure 5.1 – Main steps on the proof of Theorem 5.1.5.

Remark 5.1.1. *The condition $Lu \in \mathscr{C}(\Gamma)$ is necessary. Indeed we want to construct a strongly continuous semigroup on $\mathscr{C}(\Gamma)$ hence we need to have $Lu \in \mathscr{C}(\Gamma)$ but the fact that $D(L) \subset \mathscr{C}^2(\Gamma)$ is not sufficient for this to hold because the derivatives of $u \in \mathscr{C}^k(\Gamma)$ are in general not continuous at junctions. Therefore we only know that $Lu \in PC(\Gamma)$ for $u \in \mathscr{C}^2(\Gamma)$.*

Lemma 5.1.2. *The domain $D(L)$ is dense in $\mathscr{C}(\Gamma)$.*

Proof. See Proof A.5.1. □

Lemma 5.1.3. *The unbounded linear operator $L : D(L) \to \mathscr{C}(\Gamma)$ is closed.*

Proof. See Proof A.5.2. □

Lemma 5.1.4. *The resolvent set $\rho(L)$ of L satisfies $(0, \infty) \subset \rho(L)$ and the resolvent satisfies*

$$\|R_\lambda\|_{\mathscr{L}(\mathscr{C}(\Gamma))} \leq \frac{1}{\lambda}$$

for every $\lambda > 0$.

Proof. See Proof A.5.3. □

We are now able to prove the result first stated in [FW93, Theorem 3.1].

Theorem 5.1.5. *The operator L is the infinitesimal generator of a strongly continuous semigroup of contraction on $\mathscr{C}(\Gamma)$ which is uniquely determined by the Kirchhoff condition.*

Proof. The lemmas 5.1.2, 5.1.3 and 5.1.4 show that the operator L satisfies the conditions of the Hille-Yosida theorem (Theorem B.1.8) and thus it is the infinitesimal generator of a strongly continuous semigroup of contraction on $\mathscr{C}(\Gamma)$. The fact that the semigroup is uniquely determined by the operator L and the Kirchhoff conditions is a consequence of Proposition B.1.5. □

Consider now $(T_t)_{t\geq 0}$ the semigroup of Theorem 5.1.5. For every $x \in \Gamma$ and $t \geq 0$ we can define the linear form on $\mathscr{C}(\Gamma)$

$$\phi_x : \mathscr{C}(\Gamma) \ni f \mapsto T_t f(x).$$

Notice, that because $(T_t)_{t\geq 0}$ is a strongly continuous semigroup of contractions on $\mathscr{C}(\Gamma)$, the linear form ϕ_x is continuous and $\|\phi_x\|_{\mathscr{L}(\mathscr{C}(\Gamma),\mathbb{R})} \leq 1$. According to Riesz's representation theorem (see [Bog07, Theorem 7.10.4]) there exists a measure $p_t(x, \cdot)$ on Γ such that

$$T_t f(x) = \int_\Gamma f(y) p_t(x, dy).$$

In Proposition 3.2.10 we have proved that for every $f \in \mathscr{C}(\Gamma)$ such that $f \geq 0$ on Γ we have $R_\lambda f = (\lambda I - L)^{-1} f \geq 0$ on Γ. Then Theorem B.1.10 implies that $T_t f \geq 0$ on Γ. This shows that each measure $p_t(x, \cdot)$ must be positive and the fact that $(T_t)_{t\geq 0}$ is a semigroup of contractions imposes to each measure to have a total mass equal to one. Indeed let $\mathbf{1}$ be the constant function taking the value 1 everywhere on Γ, then clearly $\mathbf{1} \in \mathscr{C}^2(\Gamma)$ and $\partial_\alpha \mathbf{1}(v_i) = 0$ for every $\alpha \in \mathcal{A}_i$ and $i \in \mathcal{I}$. Hence f satisfies the Kirchhoff condition and $\mathbf{1} \in D(L)$. Then

$$\int_\Gamma p_t(x, dy) = \int_\Gamma \mathbf{1}(y) p_t(x, dy) = |T_t \mathbf{1}(x)| \leq 1.$$

And in fact as $\lambda R_\lambda \mathbf{1} = \mathbf{1}$ for every $\lambda > 0$ by Theorem B.1.10 we have

$$\int_\Gamma p_t(x, dy) = |T_t \mathbf{1}(x)| = 1.$$

Hence we have proved that each $p_t(x, \cdot)$ is a probability measure on $(\Gamma, \mathscr{B}(\Gamma))$. We recall the following definition.

Definition 5.1.6. By a Feller semigroup $(T_t)_{t\geq 0}$ on a Polish space E we mean strongly continuous semigroup on $\mathscr{C}_0(E)$, the space of continuous functions vanishing at infinity, such that $T_t f \geq 0$ if $f \geq 0$ and $\|T_t\|_{\mathscr{L}(\mathscr{C}_0(E))} \leq 1$. $\triangleleft$

In particular the semigroup we just built is Feller. Moreover if the family of linear operators

$$\mathscr{C}_0(E) \ni f \mapsto T_t f(\cdot) = \int_E f(y) p_t(\cdot, dy),$$

where $p_t(\cdot, \cdot)$ is the transition function of some Markov process, is a Feller semigroup we say that the process is a Feller process. The following standard result is taken from [EK86, Theorem 2.7 p.169].

Theorem 5.1.7. *Let E be a locally compact and separable metric space and let $(T_t)_{t\geq 0}$ be a Feller semigroup on $\mathscr{C}_0(E)$ the space of continuous functions vanishing at infinity. Then for each probability measure ν there exists a Markov process X corresponding to $(T_t)_{t\geq 0}$ with initial distribution ν in the sense that*

$$\mathbb{E}_x [f(X_t)] = T_t f(x)$$

for every $f \in D(L)$, $t \geq 0$ and $x \in \Gamma$.

As Γ is a compact Polish space (note that the boundedness of Γ implies that $\mathscr{C}_0(\Gamma) = \mathscr{C}(\Gamma)$) we can indeed apply the theorem above and we can consider a Markov process X on Γ whose associated semigroup is $(T_t)_{t\geq 0}$ and whose transition probabilities are given by $p_t(x, \cdot)$ for every $t > 0$ and $x \in \Gamma$. Being a Feller process it has a càdlàg modification[1] (see [RY99, Theorem 2.7 p.91]) which we also denote X and from now on we will only consider this modification of the process. To prove that X has in fact continuous paths we will use the following result which combines [Tai20, Theorem 3.33] and [Tai20, Theorem 15.9].

Theorem 5.1.8. *Let $(T_t)_{t\geq 0}$ is a Feller semigroup on Γ and X is its associated càdlàg Markov process with transition function p_t. If the infinitesimal generator L of $(T_t)_{t\geq 0}$ is such that for every $\epsilon > 0$ and $x \in \Gamma$ there exists a function $f \in D(L)$ such that*

1. *$f(x) \geq 0$ on Γ,*
2. *$f(y) > 0$ for $y \in \Gamma \setminus B(x, \epsilon)$,*
3. *$f(z) = Lf(z) = 0$ on a neighborhood of x,*

then X has a.s. continuous sample paths.

In order to prove the continuity of the sample paths of X we have to to check that the transition function obtained for X satisfies the criterion of Theorem 5.1.8. Assume first that x lies in the interior of an edge Γ_α. Then we know, using a form of cutoff function, that we can find a function f such that $f_\alpha \in \mathscr{C}^\infty([0, \ell_\alpha])$ for every $\alpha \in \mathcal{A}$ satisfying the assumptions of Theorem 5.1.8 (see [Hö3, Theorem 1.4.1]). Moreover we can choose this function such that it is constant and equal to one in the neighborhood of every vertex so that $f \in D(L)$. If x is a vertex the argument is similar, we just construct a cutoff function on each adjacent edge of the vertex.

Corollary 5.1.9. *The sample paths of the Markov process X on Γ generated by L are almost surely continuous.*

We end this section by recalling a standard result form the theory of Markov processes (see [RY99, Proposition 1.6 p.284]) which can be seen as a weak form of Itô's formula and will be used in the next section to derive the stationary Fokker-Plank-Kolmogorov equation.

Proposition 5.1.10 (Martingale property)**.** *If $f \in D(L)$ then the process*

$$M_t^f = f(X_t) - f(X_0) - \int_0^t Lf(X_s)\, ds$$

is a martingale. In particular we have

$$\mathbb{E}_x\left[f(X_t)\right] = f(x) + \mathbb{E}_x\left[\int_0^t Lf(X_s)\, ds\right].$$

[1] A stochastic process Y is said to be a modification of the process X defined on the same probability space if

$$Y_t = X_t \quad \text{a.s.}$$

for every t.

5.2 The stationary Fokker-Plank-Kolmogorov equation

This section is devoted to the derivation of the Fokker-Plank-Kolmogorov equation associated to the invariant measure.

Let $(T_t)_{t\geq 0}$ be a Feller semigroup associated to Markov transition probabilities $p_t(x,\cdot)$ on Γ.

Definition 5.2.1. An invariant measure for $(T_t)_{t\geq 0}$ is a probability measure μ such that for every continuous function f on Γ one has

$$\mathbb{E}_\mu\left[f(X_t)\right] := \int_\Gamma T_t f(x)\mu(dx) = \int_\Gamma f(x)\mu(dx).$$

$\triangleleft$

We want to prove that the process X obtained above admits an invariant measure. This is a consequence of the following theorem which can be found in [DPZ96].

Theorem 5.2.2 (Krylov-Bogoliubov). *Let E be a Polish space, $(T_t)_{t\geq 0}$ be a Feller semigroup on E with transition probabilities $p_t(x,\cdot)$ and μ be a probability measure on E. Define the probability measures*

$$R_T(x,A) = \frac{1}{T}\int_0^T p_t(x,A)\,dt \quad \forall A \in \mathcal{B}(E)$$

for every $x \in E$ and $T > 0$. Define another family of probability measure

$$R_T^\star\mu(A) = \int_E R_T(x,A)\mu(dx) \quad \forall A \in \mathcal{B}(E)$$

for every $T > 0$. If the family $(R_T^\star\mu)_{T>0}$ is uniformly tight then there exists an invariant measure for $(T_t)_{t\geq 0}$.

Corollary 5.2.3. *The Feller process X defined on Γ admits an invariant measure.*

Proof. This follows from the fact that any family of probability measures on a compact topological space is uniformly tight. One may for example choose μ to be the initial distribution of the Feller process in Theorem 5.2.2. $\square$

We are now in position to derive the stationary FPK equation for our class of stochastic processes. Consider any $u \in D(L)$ and let μ be an invariant measure of the process X generated by L. Then according to Proposition 5.1.10 we have

$$\mathbb{E}_\mu[u(X_0)] - \mathbb{E}_\mu\left[u(X_t)\right] + \mathbb{E}_\mu\left[\int_0^t Lu(X_s)\,ds\right] = 0.$$

Using the fact that μ is an invariant measure this can be written as

$$\int_\Gamma u(x)\,\mu(dx) - \int_\Gamma u(x)\,\mu(dx) + \int_\Gamma\int_0^t Lu(x)\,\mu(dx)dt = 0$$

which yields

$$\int_\Gamma Lu(x)\,\mu(dx) = 0 \quad \text{for every } u \in D(L).$$

This leads to the following definition.

Definition 5.2.4. We say that a measure μ satisfies the equation

$$L^\star \mu = 0$$

where $L^\star$ is the formal adjoint[2] of L if for every $u \in D(L)$ we have

$$\int_\Gamma Lu(x)\,\mu(dx) = 0.$$

◁

In order to study the FPK equation we prove that any invariant measure μ admits a density with respect to $\mathscr{L}$. The proof follows the argument made in [BKR01, Theorem 2.1] which was used to prove an analogue result for FPK equations on domains inside $\mathbb{R}^d$ (see also [BKRS15, Corollary 1.5.3]).

Theorem 5.2.5 (Bogachev-Krylov-Röckner)**.** *Let μ be a positive finite Borel measure satisfying*

$$L^\star \mu = 0.$$

Then μ admits a density m with respect to $\mathscr{L}$. In particular any invariant measure of X has a density.

Proof. Let $\lambda > 0$, then for every $v \in D(L)$ we have

$$\int_\Gamma (-L + \lambda I)v(x)\,\mu(dx) = \lambda \int_\Gamma v(x)\,\mu(dx).$$

Furthermore according to Theorem 3.2.9 and Proposition 3.2.6, for every $f \in \mathscr{C}^\infty(\Gamma)$ there exists $u \in D(L)$ such that $(-L+\lambda I)u(x) = f(x)$ for every $x \in \Gamma \setminus \mathcal{V}$ and this can be extended by continuity to the whole of Γ. Moreover the function u satisfies $\|u\|_{C^2(\Gamma)} \le C\|f\|_{L^2(\Gamma)}$. This yields

$$\begin{aligned}\left|\int_\Gamma f(x)\,\mu(dx)\right| &\le \lambda \int_\Gamma |u|(x)\,\mu(dx) \le \lambda\mu(\Gamma)\|u\|_{\mathscr{C}^2(\Gamma)} \\ &\le C\lambda\mu(\Gamma)\|f\|_{L^2(\Gamma)}.\end{aligned}$$

Now choose any $A \in \mathcal{B}(\Gamma)$. By convolution of χ_A with a sequence of mollifiers we obtain a sequence $(f_n)_{n\in\mathbb{N}}$ in $PC(\Gamma)$ such that $f_{n,\alpha} \in \mathscr{C}^\infty([0,\ell_\alpha])$ for every $\alpha \in \mathcal{A}$ and converging to χ_A in $L^2(\Gamma)$. The sequence is also uniformly bounded in $L^\infty(\Gamma)$. We have

$$\left|\int_\Gamma f_n(x)\,\mu(dx)\right| \le C\lambda\mu(\Gamma)\int_\Gamma |f_n(x)|^2\,dx$$

[2]The rigorous definition of the adjoint operator will be obtained later.

for every $n \in \mathbb{N}$. As $(f_n)_{n\in\mathbb{N}}$ converges in $L^2(\Gamma)$ and is uniformly bounded in $L^\infty(\Gamma)$ we can apply the dominated convergence theorem to obtain

$$\mu(A) \leq C\lambda\mu(\Gamma)\,\mathscr{L}(A)$$

which proves that μ is absolutely continuous with respect to the measure $\mathscr{L}$. We can now apply the Radon-Nikodym theorem (see [Bog07, Theorem 3.2.2]) which states that there exists an integrable function $m : \Gamma \to \mathbb{R}$ such that

$$\mu(A) = \int_A m(x)\, dx$$

for every $A \in \mathcal{B}(\Gamma)$.

□

We can now continue our derivation of the FPK equation, we extend the argument used in [ADLT19]. For every $u \in D(L)$, we have that

$$\int_\Gamma Lu(x)m(x)\, dx = 0.$$

It will be convenient to write L in divergence form :

$$Lu = \partial(\sigma^2 \partial u) + \tilde{b}\partial u$$

where $\tilde{b} = b - 2\sigma\partial\sigma$. Then we have

$$\sum_{\alpha\in\mathcal{A}} \int_0^{\ell_\alpha} \Big(\partial(\sigma_\alpha^2(s)\partial u_\alpha(s)) + \tilde{b}_\alpha(s)\partial u_\alpha(s)\Big)\, m_\alpha(s)\, ds = 0 \tag{5.1}$$

which implies that

$$\partial(-\sigma_\alpha^2 \partial m_\alpha + \tilde{b}_\alpha m_\alpha) = 0 \tag{5.2}$$

in the sense of distributions for every $\alpha \in \mathcal{A}$. From a standard result of the theory of distributions (see [Hö3, Theorem 3.1.4]) we deduce that there exist constants $(c_\alpha)_{\alpha\in\mathcal{A}}$ such that

$$-\sigma_\alpha^2 \partial m_\alpha + \tilde{b}_\alpha m_\alpha = c_\alpha$$

for every $\alpha \in \mathcal{A}$. This implies that $m_\alpha \in W^{1,1}(0, \ell_\alpha)$ and the standard Sobolev inequality together with a bootstrap argument gives $m_\alpha \in \mathscr{C}^1([0, \ell_\alpha])$.

As m is smooth enough we can integrate by parts

$$\begin{aligned}
0 = \int_\Gamma Lu(x)m(x)\, dx &= \sum_{\alpha\in\mathcal{A}} \int_0^{\ell_\alpha} \Big[\partial(\sigma_\alpha^2(s)\partial u_\alpha(s))m_\alpha(s) + \tilde{b}_\alpha(s)\partial u_\alpha(s)m_\alpha(s)\Big]\, ds \\
&= \sum_{i\in\mathcal{I}} \sum_{\alpha\in\mathcal{A}_i} \sigma^2_{|\Gamma_\alpha}(v_i) m_{|\Gamma_\alpha}(v_i) \partial_\alpha u(v_i) + \sum_{\alpha\in\mathcal{A}} \int_0^{\ell_\alpha} \Big[\partial u_\alpha(s)\Big(-\sigma_\alpha^2 \partial m_\alpha(s) + \tilde{b}_\alpha(s) m_\alpha(s)\Big)\Big]\, ds \\
&= \sum_{i\in\mathcal{I}} \sum_{\alpha\in\mathcal{A}_i} \sigma^2_{|\Gamma_\alpha}(v_i) m_{|\Gamma_\alpha}(v_i) \partial_\alpha u(v_i) + \sum_{\alpha\in\mathcal{A}} c_\alpha \int_{\Gamma_\alpha} \partial u(x)\, dx
\end{aligned}$$

Choose any $i \in \mathcal{I}$ and consider a test function $u \in D(L)$ of the form

$$\begin{cases} u(v_j) = \delta_{i,j}, & \forall j \in \mathcal{I}, \\ \partial_\alpha u(v_j) = 0, & \forall j \in \mathcal{I},\ \forall \alpha \in \mathcal{A}_i. \end{cases}$$

This gives

$$0 = \sum_{\alpha \in \mathcal{A}} c_\alpha \int_0^{\ell_\alpha} \partial u_\alpha(s)\, ds = \sum_{\alpha \in \mathcal{A}_i} n_{i,\alpha} c_\alpha. \tag{5.3}$$

where the numbers $n_{i,\alpha}$ are those defined in (2.3). Now choose $\alpha, \beta \in \mathcal{A}_i$ and consider a test function $u \in D(L)$ such that

1. u takes the same value at each vertex of Γ and $\int_{\Gamma_\delta} \partial u(x)\, dx = 0$ for every $\delta \in \mathcal{A}$,
2. $\partial_\alpha u(v_i) = \frac{1}{p_{i\,\alpha}}$, $\partial_\beta u(v_i) = \frac{-1}{p_{i,\beta}}$ and all other directional derivatives take the value 0.

With test-functions of this form we find that

$$\frac{m_{|\Gamma_\alpha}(v_i)}{\gamma_{i,\alpha}} = \frac{m_{|\Gamma_\beta}(v_i)}{\gamma_{i,\beta}}$$

for every $i \in \mathcal{I}$ and $\alpha, \beta \in \mathcal{A}_i$. Then for every $i \in \mathcal{I}$ and each $\alpha \in \mathcal{A}_i$ we can multiply (5.2) by $n_{i,\alpha}$ and taking the sum over $\mathcal{A}_i$. This yields using (5.3)

$$\begin{aligned} 0 &= \sum_{\alpha \in \mathcal{A}_i} -\sigma^2_{|\Gamma_\alpha} \partial_\alpha m_{|\Gamma_\alpha}(v_i) + n_{i,\alpha} \left(m_{|\Gamma_\alpha}(v_i) \tilde{b}_{|\Gamma_\alpha}(v_i) - c_\alpha \right) \\ &= \sum_{\alpha \in \mathcal{A}_i} -\sigma^2_{|\Gamma_\alpha} \partial_\alpha m_{|\Gamma_\alpha}(v_i) + n_{i,\alpha} m_{|\Gamma_\alpha}(v_i) \tilde{b}_{|\Gamma_\alpha}(v_i). \end{aligned}$$

Therefore the density m satisfies the following equation

$$\begin{cases} -\partial(\sigma^2 \partial m) + \partial(\tilde{b} m) = 0 & \text{on } \Gamma \setminus \mathcal{V}, \\ \sum_{\alpha \in \mathcal{A}_i} \sigma^2_{|\Gamma_\alpha} \partial_\alpha m_{|\Gamma_\alpha}(v_i) - n_{i,\alpha} m_{|\Gamma_\alpha}(v_i) \tilde{b}_{|\Gamma_\alpha}(v_i) = 0 & \forall i \in \mathcal{I}, \\ \frac{m_{|\Gamma_\alpha}(v_i)}{\gamma_{i,\alpha}} = \frac{m_{|\Gamma_\beta}(v_i)}{\gamma_{i,\beta}} & \forall i \in \mathcal{I},\ \forall \alpha, \beta \in \mathcal{A}_i. \end{cases} \tag{SFPK}$$

By Theorem 3.3.2 the problem (SFPK) as a unique weak solution $m \in \mathcal{W}$. In particular the invariant measure is unique.

Chapter 6

The Hamilton-Jacobi-Bellman equation

This section gathers some applications of the previous chapters to Hamilton-Jacobi-Bellman equations. We first make a formal derivation of the ergodic HJB equation for the optimal control of the Markov process obtained in Chapter 5. We also recall the main theorems proved in [ADLT19] for this problem. Then we present some results also obtained in [ADLT19] for the stationary mean field game system with local coupling and prove the case of non-local coupling which was left as a remark in [ADLT19]. Finally we apply a standard result from the theory of semigroups to prove a existence and uniqueness theorem for the parabolic HJB equation.

6.1 Ergodic HJB equation

Let $(\mu_\alpha)_{\alpha\in\mathcal{A}}$ be strictly positive constants and $a \in PC(\Gamma)$. In this section our goal is to derive and study the Hamilton-Jacobi-Bellman (HJB for short) equation associated to the optimal control problem of the Markov process X_t defined by its generator

$$Lu(x) = L_\alpha u(x) = \mu_\alpha \partial^2 u(x) + a_{|\Gamma_\alpha}(x)\partial u(x)$$

as in Chapter 5. The function a is the control associated to the long run average cost

$$\mathcal{J}(x,a) = \liminf_{T\to\infty} \frac{1}{T}\mathbb{E}_x\left[\int_0^T l(X_s, a(X_s))\, ds\right]$$

where we assume that the running cost is of the form $l(x,a) = l_\alpha(\pi_\alpha^{-1}(x), a)$ for $x \in \Gamma_\alpha$.

In what follows we will often write a_s for $a(X_s)$.

Remark 6.1.1. *The Markov property can be applied to the control a and in particular we have*

$$\mathbb{E}_x\left[l(X_t, a_t)\right] = \mathbb{E}_x\left[\mathbb{E}_{X_s}\left[l(X_{t-s}, a_{t-s})\right]\right]$$

for every $0 \le s < t$ whenever $l(\cdot, a(\cdot))$ is a bounded measurable function.

6.1.1 The infinite horizon problem

We are first going to consider the infinite horizon problem and and make an formal derivation of its associated Hamilton-Jacobi-Bellman equation. For every real $\lambda > 0$, define the cost functional

$$\mathcal{J}_\lambda(x,a) = \mathbb{E}_x\left[\int_0^\infty l(X_s,a_s)e^{-\lambda s}\,ds\right].$$

and its associated value function $u_\lambda(x) = \inf_a \mathcal{J}(x,a)$. We assume that a dynamic programming principle of the following form holds.

Assumption 1 (Dynamic programming principle)**.** For every $t > 0$ we have

$$u_\lambda(x) = \inf_a \mathbb{E}_x\left[\int_0^t l(X_s,a_s)e^{-\lambda s}\,ds + u_\lambda(X_t)e^{-\lambda t}\right].$$

◁

We are now able to formally derive the HJB equation associated to u_λ. We proceed similarly to [CP20]. According to the dynamic programming principle we have for a small $h > 0$

$$u_\lambda(x) = \inf_a \mathbb{E}_x\left[\int_0^h l(X_s,a_s)e^{-\lambda s}\,ds + u_\lambda(X_h)e^{-\lambda h}\right]$$

and hence

$$0 = \inf_a \mathbb{E}_x\left[\int_0^h l(X_s,a_s)e^{-\lambda s}\,ds + u_\lambda(X_h)e^{-\lambda h} - u_\lambda(x)\right]. \tag{6.1}$$

Assuming u_λ belongs to $D(L)$, in particular this implies that u_λ must satisfy the Kirchhoff condition associated with the generator, by Proposition 5.1.10 we have

$$\mathbb{E}_x\left[u_\lambda(X_h)\right] = \mathbb{E}_x\Bigg[u_\lambda(x) + \sum_{\alpha\in\mathcal{A}}\int_0^h \chi_{\{X_s\in\Gamma_\alpha\setminus\mathcal{V}\}}\left(\mu_\alpha\partial^2 u_\lambda(X_s) + a(X_s)\partial u_\lambda(X_s)\right)\,ds\Bigg].$$

Injecting this into (6.1) gives

$$\begin{aligned}0 = \inf_a \mathbb{E}_x\Bigg[&\int_0^h l(X_s,a_s)e^{-\lambda s}\,ds + (e^{-\lambda h}-1)u_\lambda(x)\\ &+ e^{-\lambda h}\sum_{\alpha\in\mathcal{A}}\int_0^h \chi_{\{X_s\in\Gamma_\alpha\setminus\mathcal{V}\}}\left(\mu_\alpha\partial^2 u_\lambda(X_s) + a(X_s)\partial u_\lambda(X_s)\right)\,ds\Bigg].\end{aligned}$$

Then dividing by h and taking the limit when h tends to 0 formally yields[1]

$$0 = \inf_a\left\{l(x,a) - \lambda u_\lambda(x) + \sum_{\alpha\in\mathcal{A}}\chi_{\{x\in\Gamma_\alpha\setminus\mathcal{V}\}}\left(\mu_\alpha\partial^2 u_\lambda(x) + a\partial u_\lambda(x)\right)\right\}.$$

[1]Note that this step implies an interversion of limits.

Finally we have obtained

$$(\mathrm{HJB}_{\lambda,h}) \qquad \begin{cases} -\mu_\alpha \partial^2 u_\lambda(x) + H(x, \partial u_\lambda) + \lambda u_\lambda(x) = 0 & \text{for } x \in \Gamma_\alpha \setminus \mathcal{V}, \\ \sum_{\alpha \in \mathcal{A}_i} p_{i,\alpha} \partial_\alpha u(v_i) = 0 & \text{for every } i \in \mathcal{I}, \\ u_{|\Gamma_\alpha}(v_i) = u_{\Gamma_\beta}(v_i) & \text{for all } \alpha, \beta \in \mathcal{A}_i, \end{cases}$$

where the Hamiltonian is given by $H(x,p) = \sup_a \{-ap - l(x,a)\}$ and for every $i \in \mathcal{I}$ the positive reals $(p_{i,\alpha})_{\alpha \in \mathcal{A}_i}$ are defined by $p_{i,\alpha} = \mu_\alpha \gamma_{i,\alpha}$ for some positive number $\gamma_{i,\alpha}$ and $\sum_{\alpha \in \mathcal{A}_i} p_{i,\alpha} = 1$.

We also assume that we have a verification theorem for this optimal control problem.

Assumption 2 (Verification)**.** Assume $v_\lambda \in \mathscr{C}^2(\Gamma)$ solves ($\mathrm{HJB}_{\lambda,h}$). Then

$$v_\lambda(x) \leq u_\lambda(x)$$

for every $x \in \Gamma$. Consequently we then have $v_\lambda = u_\lambda$, where u_λ is the value function associated with the optimal control problem. ◁

In the case where Assumptions 1 and 2 hold, we would have the following consequence.

Corollary 6.1.2. *Let u_λ be a solution of* ($\mathrm{HJB}_{\lambda,h}$)*. The function $a(x) = \partial_p H(x, \partial u_\lambda)$ is an optimal feedback for the stochastic control problem.*

We now come to the existence result which was proved in [ADLT19, Proposition 1] extending [CM16, Proposition 10]. In both cases the proof follows an argument introduced in [BMP83].

Theorem 6.1.3. *Assume $H_\alpha \in \mathscr{C}([0, \ell_\alpha] \times \mathbb{R})$ for every $\alpha \in \mathcal{A}$ and that there exists a constant C such that*

$$|H(x,p)| \leq C\left(1 + |p|^2\right), \quad \forall (x,p) \in \Gamma \times \mathbb{R}. \tag{6.2}$$

Then there exists a classical solution $u \in \mathscr{C}^2(\Gamma)$ of ($\mathrm{HJB}_{\lambda,h}$)*.*

Corollary 6.1.4. *In addition the the assumptions of Theorem 6.1.3 assume that H_α is locally Lipschitz continuous with respect to both variables for every $\alpha \in \mathcal{A}$. Then the solution u from Theorem 6.1.3 belongs to $\mathscr{C}^{2,1}(\Gamma)$.*

6.1.2 Back to the ergodic problem

In order to come back to the original problem we introduce the following cost functional

$$\mathcal{J}(x,a,T) = \mathbb{E}_x\left[\int_0^T l(X_s, a_s)\, dx\right] \tag{6.3}$$

and the associated value function

$$u(x,T) = \inf_a \mathcal{J}(x,a,T). \tag{6.4}$$

The reason for our study of the infinite horizon problem is the following assumption (which was proved in a different setting in [AL98, Proposition VI.1]) which makes the connection between the value function of the infinite horizon problem and the value function of the finite horizon problem.

Assumption 3. The function λu_λ converges uniformly and λ tends to 0_+ to a constant if, and only if, then function $\frac{1}{T}u(\cdot, T)$ converges uniformly as T tends to $+\infty$ to the same constant. $\triangleleft$

Therefore the study of the problem with long rung average cost is equivalent to taking to limit $\lambda \to 0_+$ in the infinite horizon problem.

Assumption 4. We assume that for every $\alpha \in \mathcal{A}$ we have

1. $H_\alpha \in \mathscr{C}^1([0, \ell_\alpha] \times \mathbb{R})$,
2. H_α is convex in p for every $x \in [0, \ell_\alpha]$,
3. there exists positive constants C_0, C_1 and some $q \in (1, 2]$ such that $H(x,p) \geq C_0|p|^q - C_1$ for $(x,p) \in [0, \ell_\alpha] \times \mathbb{R}$.

$\triangleleft$

The following theorem was obtained in [ADLT19, Theorem 3.4] extending [CM16, Theorem 4] to more general Kirchhoff conditions.

Theorem 6.1.5. *Under Assumption 4 for every $f \in PC(\Gamma)$ there exists a unique pair $(u, \rho) \in \mathscr{C}^2(\Gamma) \times \mathbb{R}$ solution of*

$$\begin{cases} -\mu_\alpha \partial^2 v + H(x, \partial v) + \rho = f & x \in \Gamma_\alpha \setminus \mathcal{V} \\ \sum_{\alpha \in \mathcal{A}_i} p_{i,\alpha} \partial_\alpha v(v_i) = 0 & i \in \mathcal{I}, \\ u_{|\Gamma_\alpha}(v_i) = u_{|\Gamma_\beta}(v_i) & \forall \alpha, \beta \in \mathcal{A}_i, \ \forall i \in \mathcal{I}, \\ \int_\Gamma u \, dx = 0. \end{cases} \tag{HJB$_e$}$$

and there exists a positive constant C such that

$$\|u\|_{\mathscr{C}^2(\Gamma)} \leq C.$$

Furthermore if the function f satisfies $f_\alpha \in \mathscr{C}^{0,\theta}([0, \ell_\alpha])$ for every $\alpha \in \mathcal{A}$, then we also have

$$\|u\|_{\mathscr{C}^{2,\theta}(\Gamma)} \leq C.$$

6.1.3 Stationary mean field games

In this section we study the mean field game system where the state space is a network. The main idea behind mean field games is to consider a continuum of identical agents which we represent by a stochastic process of the form studied in Chapter 5. It was proved in [FS00] that such a process can be interpreted as solving the stochastic differential equation

$$dX_t = \sigma(X_t)dW_t + b(X_t)dt$$

on each edge for some Brownian motion W. It is therefore natural to make the assumption that each agent can control the drift of this dynamic and hence we assume that a representative agent is a stochastic process generated by the differential operator

$$\begin{cases} Lu = \mu\partial^2 u + a\partial u \\ \sum_{\alpha\in\mathcal{A}_i} \gamma_{i,\alpha}\mu_\alpha\partial_\alpha u(v_i) = 0 \end{cases}$$

where μ is positive and constant on each edge, $a \in PC(\Gamma)$ will be the control and the constants $\gamma_{i,\alpha}$ are chosen so that $\sum_{\alpha\in\mathcal{A}_i} \gamma_{i,\alpha}\mu_\alpha = 1$ for every $i \in \mathcal{I}$.

For the stationary problem each agent uses the control a to minimize the long run average cost

$$\mathcal{J}(x,a) = \liminf_{T\to\infty} \frac{1}{T}\mathbb{E}_x\left[\int_0^T l(X_s, a(X_s)) + V[m(\cdot,s)](X_s)\, ds\right]$$

where $V : \mathcal{P}(\Gamma) \to \mathscr{C}^2(\Gamma)$ is a coupling operator and m is the density of agents. We assume that the running cost is of the form $l(x,a) = l_\alpha(\pi_\alpha^{-1}(x), a)$ for $x \in \Gamma_\alpha$. In Chapter 6 this optimal control problem was associated with the following ergodic Hamilton-Jacobi-Bellman equation

$$\begin{cases} -\mu\partial^2 u + H(x,\partial u) + \rho = V[m](x) & x \in \Gamma_\alpha \setminus \mathcal{V} \\ u_{|\Gamma_\alpha}(v_i) = u_{|\Gamma_\beta}(v_i) & \forall \alpha,\beta \in \mathcal{A}_i,\ \forall i \in \mathcal{I}, \\ \sum_{\alpha\in\mathcal{A}_i} \gamma_{i,\alpha}\mu_\alpha\partial_\alpha u(v_i) = 0 & i \in \mathcal{I}, \\ \int_\Gamma u(x)\, dx = 0. \end{cases} \tag{6.5}$$

Where the Hamiltonian is given by

$$H(x,p) = \sup_a \left\{-ap - l(x,a)\right\}.$$

Moreover we saw that for this problem the optimal feedback was given by

$$a^\star(x) = -\partial_p H(x, \partial u).$$

Therefore the generator of the stochastic process representing an agent is

$$Lv = \mu\partial^2 v(x) - \partial_p H(x,\partial u)\partial v$$

and we saw in Chapter 5 that the Fokker-Planck-Kolmogorov equation describing the stationary measure of this process was

$$\begin{cases} -\mu\partial^2 m - \partial(m\partial_p H(x,\partial u)) = 0 & \text{on } \Gamma \setminus \mathcal{V}, \\ \sum_{\alpha\in\mathcal{A}_i} \mu_\alpha\partial_\alpha m_{|\Gamma_\alpha}(v_i) + n_{i,\alpha} m_{|\Gamma_\alpha}(v_i)\partial_p H_{|\Gamma_\alpha}(v_i, \partial u_{|\Gamma_\alpha}(v_i)) = 0 & \forall i \in \mathcal{I}, \\ \frac{m_{|\Gamma_\alpha}(v_i)}{\gamma_{i,\alpha}} = \frac{m_{|\Gamma_\beta}(v_i)}{\gamma_{i,\beta}} & \forall i \in \mathcal{I},\ \forall \alpha,\beta \in \mathcal{A}_i, \\ m \geq 0, \quad \int_\Gamma m(x)\, dx = 1. \end{cases} \tag{6.6}$$

We have obtained the following mean field game system describing the coupled dynamic (SMFG)

$$\begin{cases} -\mu\partial^2 u + H(x,\partial u) + \rho = V[m](x) & x \in \Gamma\setminus\mathcal{V} \\ \mu\partial^2 m + \partial(m\partial_p H(x,\partial u)) = 0 & x\in\Gamma\setminus\mathcal{V}, \\ u_{|\Gamma_\alpha}(v_i) = u_{|\Gamma_\beta}(v_i) & \forall\alpha,\beta\in\mathcal{A}_i,\ \forall i\in\mathcal{I}, \\ \sum_{\alpha\in\mathcal{A}_i}\gamma_{i,\alpha}\mu_\alpha\partial_\alpha u(v_i) = 0 & i\in\mathcal{I}, \\ \sum_{\alpha\in\mathcal{A}_i}\mu_\alpha\partial_\alpha m_{|\Gamma_\alpha}(v_i) + n_{i,\alpha} m_{|\Gamma_\alpha}(v_i)\partial_p H_{|\Gamma_\alpha}(v_i,\partial u_{|\Gamma_\alpha}(v_i)) = 0 & \forall i\in\mathcal{I}, \\ \frac{m_{|\Gamma_\alpha}(v_i)}{\gamma_{i,\alpha}} = \frac{m_{|\Gamma_\beta}(v_i)}{\gamma_{i,\beta}} & \forall i\in\mathcal{I},\ \forall\alpha,\beta\in\mathcal{A}_i, \\ m\geq 0, \quad \int_\Gamma m(x)\,dx = 1, \quad \int_\Gamma u(x)\,dx = 0. \end{cases}$$

We now list the assumptions made on the structure of the problem.

1. For the Hamiltonian $H:\Gamma\times\mathbb{R}\to\mathbb{R}$ we assume that there exists constants $C_0, C_1, C_2 \in (0,+\infty)$ and $q\in(1,2]$ such that for every $\alpha\in\mathcal{A}$ we have
   - $H_\alpha\in\mathscr{C}^1([0,\ell_\alpha]\times\mathbb{R})$,
   - $H_\alpha(x,p)$ is convex in p for every $x\in[0,\ell_\alpha]$,
   - $H_\alpha(x,p)\geq C_0|p|^q - C_1$ for $(x,p)\in[0,\ell_\alpha]\times\mathbb{R}$,
   - $|\partial_p H_\alpha(x,p)|\leq C_2\left(|p|^{q-1}+1\right)$ for $(x,p)\in[0,\ell_\alpha]\times\mathbb{R}$.
2. The coupling operator $V:\mathcal{P}(\Gamma)\to\mathscr{C}^2(\Gamma)$ is of the form
   $$V[\tilde{m}](x) = F(m(x))$$
   with $F\in\mathscr{C}([0,+\infty))$ for every probability measure $\tilde{m}$ which is absolutely continuous with respect to the measure $\mathscr{L}$ with density m. We also assume that
   $$F(r)\geq -M$$
   for some positive constant M.

We now state the main result of [ADLT19] which generalizes [CM16, Theorem 1].

Theorem 6.1.6. *Under the running assumptions there exists a solution*

$$(u,m,\rho)\in\mathscr{C}^2(\Gamma)\times\mathcal{W}\times\mathbb{R}$$

of (SMFG)*. Furthermore, if F is locally Lipschitz continuous, then the $u\in\mathscr{C}^{2,1}(\Gamma)$. Finally if F is strictly increasing, then the solution is unique.*

We now prove the analogue result for non-local regularizing coupling, which means that we assume that the coupling operator V is such that

$$V\in\mathscr{C}(\mathcal{P}(\Gamma),\mathcal{F}_\gamma)$$

for some $\gamma>0$ and is bounded where

$$\mathcal{F}_\gamma = \left\{f:\Gamma\to\mathbb{R}:\ f_\alpha\in\mathscr{C}^{0,\gamma}([0,\ell_\alpha])\right\}.$$

The following result is announced in [ADLT19, Remark 12] and as stated there the proof is very similar to the one for [ADLT19, Theorem 4.1].

Theorem 6.1.7. *Under the previous assumptions on H and the non-local regularizing assumption on V there exists a solution $(u, m, \rho) \in \mathscr{C}^{2,\gamma}(\Gamma) \times \mathcal{W} \times \mathbb{R}$ of (SMFG).*

Proof. For $\sigma \in (0, 1/2)$ we define the following function space

$$\mathcal{M}_\sigma = \left\{ m : \Gamma \to \mathbb{R} : \ m_\alpha \in \mathscr{C}^{0,\sigma}([0, \ell_\alpha]) \text{ and } \frac{m_{|\Gamma_\alpha}(v_i)}{\gamma_{i,\alpha}} = \frac{m_{|\Gamma_\beta}(v_i)}{\gamma_{i,\beta}}, \ \forall \alpha, \beta \in \mathcal{A}_i, \ \forall i \in \mathcal{I} \right\}.$$

Provided with the following norm

$$\|m\|_{\mathcal{M}_\sigma} = \|m\|_{L^\infty(\Gamma)} + \max_{\alpha \in \mathcal{A}} \|m_\alpha\|_{\mathscr{C}^{0,\sigma}([0,\ell_\alpha])}$$

the space $\mathcal{M}_\sigma$ becomes a Banach space. We also consider the following closed and convex subset of $\mathcal{M}_\sigma$

$$\mathcal{K} = \left\{ m \in \mathcal{M}_\sigma : \ m \geq 0 \text{ and } \int_\Gamma m \, dx = 1 \right\}.$$

As is usually done for mean field game systems we are going to apply Schauder's fixed point theorem. Let $m \in \mathcal{K}$, then from the assumption on V we know that $V[m]$ belongs to $\mathcal{F}_\gamma$ and according to Theorem 6.1.5 there exists a unique solution $(u, \rho) \in \mathscr{C}^{2,\gamma}(\Gamma) \times \mathbb{R}$ of

$$\begin{cases} -\mu_\alpha \partial^2 u + H(x, \partial u) + \rho = V[m] & x \in \Gamma_\alpha \setminus \mathcal{V}, \\ \sum_{\alpha \in \mathcal{A}_i} p_{i,\alpha} \partial_\alpha v(v_i) = 0 & i \in \mathcal{I}, \\ \int_\Gamma u(x) \, dx = 0. \end{cases} \tag{6.7}$$

Then from Theorem 3.3.2 there also exists a unique solution $\bar{m} \in \mathcal{K} \cap W$ of

$$\begin{cases} -\mu \partial^2 m + \partial(m \partial_p H(x, \partial u)) = 0 & \text{on } \Gamma \setminus \mathcal{V}, \\ \sum_{\alpha \in \mathcal{A}_i} \sigma^2_{|\Gamma_\alpha} \partial_\alpha m_{|\Gamma_\alpha}(v_i) - n_{i,\alpha} m_{|\Gamma_\alpha}(v_i) b'_{|\Gamma_\alpha}(v_i) = 0 & \forall i \in \mathcal{I}, \\ \frac{m_{|\Gamma_\alpha}(v_i)}{\gamma_{i,\alpha}} = \frac{m_{|\Gamma_\beta}(v_i)}{\gamma_{i,\beta}} & \forall i \in \mathcal{I}, \ \forall \alpha, \beta \in \mathcal{A}_i \\ m \geq 0, \quad \int_\Gamma m \, dx = 1. \end{cases} \tag{6.8}$$

Hence the application $T : \mathcal{K} \to \mathcal{K}$ defined by $T(m) = \bar{m}$ is well defined.

We first claim that T *is continuous*. Indeed let $m_n, m \in \mathcal{K}$ for $m \in \mathbb{N}$ such that m_n converges to m in $\mathcal{M}_\sigma$ and define $\bar{m}_n = T(m_n)$, $\bar{m} = T(m)$. We prove that $\bar{m}_n$ converges to $\bar{m}$ in $\mathcal{M}_\sigma$. Indeed for every $n \in \mathbb{N}$ there exists a unique solution $(u_n, \rho_n) \in \mathscr{C}^{2,\gamma}(\Gamma) \times \mathbb{R}$ of the associated HJB problem (6.7) associated to m_n and there also exists a unique solution $(u, \rho) \in \mathscr{C}^{2,\gamma}(\Gamma) \times \mathbb{R}$ associated to m. As the image of V is assumed to be bounded in $\mathcal{F}_\gamma$ we deduce that (u_n, ρ_n) is bounded in $\mathscr{C}^{2,\gamma}(\Gamma) \times \mathbb{R}$, uniformly in n. Therefore by Ascoli's theorem there exists some $\bar{u} \in \mathscr{C}^2(\Gamma)$ such that u_n converges to $\bar{u}$ in $\mathscr{C}^2(\Gamma)$. Moreover there also exists some $\bar{\rho} \in \mathbb{R}$ such that ρ_n converges to $\bar{\rho}$. We have enough regularity to pass to the limit $n \to \infty$ in (6.7) and we deduce that $(\bar{u}, \bar{\rho})$ that is a solution of (6.7) associated to m. By the uniqueness of solution of (6.7) we have $(\bar{u}, \bar{\rho}) = (u, \rho)$.

We can now consider $\bar{m}_n = T(m_n)$ and $\bar{m} = T(m)$. From the convergence of u_n to u in $\mathscr{C}^2(\Gamma)$ we deduce that $\partial_p H(\cdot, \partial u_n)$ converges to $\partial_p H(\cdot, \partial u)$ in $PC(\Gamma)$. In particular the

$\partial_p H(\cdot, \partial u_n)$ are bounded in $PC(\Gamma)$ and therefore the $\bar{m}_n$ are bounded in W. Then, on the one hand we have, from the reflexivity of W, that $\bar{m}_n$ converges weakly to some $\hat{m}$ in W and on the other hand, from the compact embedding of W into $\mathcal{M}_\sigma$ for $\sigma \in (0, 1/2)$, that $\bar{m}_n$ converges strongly to $\hat{m}$ in $\mathcal{M}_\sigma$. From the weak converges we deduce that $\hat{m}$ is a weak solution of (6.8) associated to u. This solution being unique we have $\hat{m} = \bar{m}$. We have proved the continuity of T.

We also claim that $T(\mathcal{K})$ *is precompact* in $\mathcal{M}_\sigma$. Indeed from the assumption on V we know that $V[\mathcal{K}]$ is bounded in $\mathcal{F}_\gamma$. This implies that $T[\mathcal{K}]$ is bounded in W. As before we conclude with to compact embedding $W \hookrightarrow \mathcal{M}_\sigma$.

Finally we can apply Schauder's fixed point theorem (see [GT01, Corollary 11.2]) to conclude that T admits a fixed point in $\mathcal{K}$ which must then be a solution of (SMFG).

□

Following [LL07, Theorem 2.4] we can impose a monotony condition on the coupling operator to obtain a uniqueness result.

Theorem 6.1.8. *In addition to the assumptions of Theorem 6.1.7 suppose that the coupling operator V satisfies the Lasry-Lions monotony condition*

$$\int_\Gamma (V[m_1] - V[m_2])\,(m_2 - m_1)(dx) \geq 0 \quad \forall m_1, m_2 \in \mathcal{P}(\Gamma). \tag{6.9}$$

Then there exists at most one solution (SMFG)*.*

Proof. For simplicity, and because it is the case of interest for us here, in the rest of the proof we assume that the probability measures have a density with respect to $\mathscr{L}$. Let (u_1, m_1, ρ_1) and (u_2, m_2, ρ_2) are two solutions of (SMFG). Then integrating against $(m_1 - m_2)$ in the HJB equation (6.7) satisfied by (u_i, ρ_i) for $i = 1, 2$ and taking the difference of the two equations we find

$$\begin{aligned}\int_\Gamma \left[-\mu(\partial^2 u_1 - \partial^2 u_2) + H(x, \partial u_1) - H(x, \partial u_2) + \rho_1 - \rho_2\right](m_1 - m_2)dx \\ = \int_\Gamma (V[m_1] - V[m_2])\,(m_1 - m_2)\,dx.\end{aligned}$$

After integration by parts and using the conditions satisfied by u_i and m_i for $i = 1, 2$ this can be rewritten

$$\begin{aligned}\int_\Gamma \mu\partial(u_1 - u_2) \cdot \partial(m_1 - m_2) + (H(x, \partial u_1) - H(x, \partial u_2)) \cdot (m_1 - m_2)\,dx + \rho_1 - \rho_2 \\ = \int_\Gamma (V[m_1] - V[m_2])\,(m_1 - m_2)\,dx.\end{aligned}$$

Then integrating against $(u_1 - u_2)$ in the FPK equation (6.8) satisfied by m_i for $i = 1, 2$ and the taking the difference, we find

$$\int_\Gamma \left[\mu(\partial^2 m_1 - \partial^2 m_2) + \partial(m_1 \partial_p H(x, \partial u_1) - m_2 \partial_p H(x, \partial u_2))\right](u_1 - u_2)\,dx = 0.$$

Again integrating by parts this can be rewritten

$$\int_\Gamma \mu\partial(m_1 - m_2)\cdot\partial(u_1-u_2) + (m_1\partial_p H(x,\partial u_1) - m_2\partial_p H(x,\partial u_2))\cdot\partial(u_1-u_2).$$

Taking the difference of the two expression we just obtained and rearranging the terms we find

$$\begin{aligned}0 = &\int_\Gamma (V[m_1]-V[m_2])\,(m_1-m_2)\,dx\\ &+\int_\Gamma m_1\left(H(x,\partial u_2)-H(x,\partial u_1)-\partial_p H(x,\partial u_1)\cdot\partial(u_2-u_1)\right)\,dx\\ &+\int_\Gamma m_2\left(H(x,\partial u_1)-H(x,\partial u_2)-\partial_p H(x,\partial u_2)\cdot\partial(u_1-u_2)\right)\,dx+\rho_2-\rho_1.\end{aligned}$$

Up to inverting the indexes we may assume that $\rho_2\geq\rho_1$. Therefore $\rho_2-\rho_1\geq 0$. Moreover the convexity of H with respect to the second variable implies that

$$H(x,\partial u_i)-H(x,\partial u_j)-\partial_p H(x,\partial u_i)\cdot\partial(u_j-u_i)\geq 0$$

and because $m_1, m_2 \geq 0$ and V satisfies the Lasry-Lions monotony condition we deduce that every term in the sum of the right-hand side is nonnegative and must hence all cancel. In particular we must have

$$\int_\Gamma (V[m_1]-V[m_2])\,(m_1-m_2)\,dx = 0.$$

It is enough to consider the case $V[m_1]=V[m_2]$. In this case the uniqueness of the solutions of the HJB part of the problem implies that $(u_1,\rho_1)=(u_2,\rho_2)$ and then by uniqueness of the solution of the FPK part we also have $m_1=m_2$. This concludes the proof.

□

6.2 Parabolic HJB equation

In this section we want to prove the existence and uniqueness of solutions to the following problem.

$$\begin{cases}-\partial_t u-\mu\partial_x^2 u+H(x,\partial_x u)=f & \text{in } (\Gamma\setminus\mathcal{V})\times(0,T),\\ u_{|\Gamma_\alpha}(t,v_i)=u_{|\Gamma_\beta}(t,v_i) & \forall\alpha,\beta\in\mathcal{A}_i,\ \forall i\in\mathcal{I},\ \forall t\in(0,T),\\ \sum_{\alpha\in\mathcal{A}_i}\gamma_{i,\alpha}\mu_\alpha\partial_\alpha u(v_i)=0 & \forall i\in\mathcal{I},\ \forall t\in(0,T),\\ u(T,\cdot)=u_T.\end{cases}\tag{6.10}$$

To achieve this we follow the steps of [Paz83, Section 8.4], see also [Hen81, Section 3.6]. This section is analogous [vBN96, Section 4] where the authors consider ramified spaces.

Assumption 5. We make the following assumptions

1. The Hamiltonian $H : \Gamma \times \mathbb{R} \to \mathbb{R}$ satisfies for every $\alpha \in \mathcal{A}$
   (a) $H_\alpha \in \mathscr{C}^1([0, \ell_\alpha] \times \mathbb{R})$,
   (b) $H_\alpha(x, \cdot)$ is convex for every $x \in [0, \ell_\alpha]$,
   (c) $H_\alpha(x, p) \leq C_0(1 + |p|)$ for every $(x, p) \in [0, \ell_\alpha] \times \mathbb{R}$,
   (d) $|\partial_p H(x, p)| \leq C_0$ for every $(x, p) \in [0, \ell_\alpha] \times \mathbb{R}$,
   (e) $|\partial_x H(x, p)| \leq C_0(1 + |p|)$ for every $(x, p) \in [0, \ell_\alpha] \times \mathbb{R}$.
2. The function f belongs to $H^1(0, T, W_b^{1,\infty}(\Gamma))$.
3. The function u_T belongs to $H^2(\Gamma)$ and satisfies the Kirchhoff condition

$$\sum_{\alpha \in \mathcal{A}_i} \gamma_{i,\alpha} \mu_\alpha \partial_\alpha v_T(v_i) = 0$$

   for every $i \in \mathcal{I}$.

◁

Considering the differential operator $L = -\mu \partial^2$ in $L^2(\Gamma)$ with domain

$$D(L) = \left\{ u \in H^2(\Gamma) : \sum_{\alpha \in \mathcal{A}_i} \gamma_{i,\alpha} \mu_\alpha \partial_\alpha u(v_i) = 0, \ \forall i \in \mathcal{I} \right\}$$

we would like to apply the results from Appendix B.3.3. However in this case L does not satisfy Assumptions 6 and 7, in particular L does not satisfy $0 \in \rho(-L)$. To solve this issue we instead consider the perturbed operator $L_0 = L + \lambda_0 I$, with $D(L_0) = D(L)$, which satisfies these assumptions for λ_0 large enough. In particular $-L_0$ is the infinitesimal generator of an analytic semigroup on $L^2(\Gamma)$. We then rewrite (6.10) as

$$\begin{cases} -\boldsymbol{u}'(t) + L_0 \boldsymbol{u}(t) = f_0(t, \boldsymbol{u}) & \forall t \in (0, T) \\ \boldsymbol{u}(T) = u_T, \end{cases} \tag{6.11}$$

where $f_0(t, u) = f(t, u) - \lambda_0 u - H(\cdot, \partial u)$. We check that this problem satisfies the assumptions of Theorem B.3.8. The main difficulty is to find the fractional power $0 < \alpha \leq 1$. The first step consists in the following proposition, which extends a standard result of real analysis to networks.

Proposition 6.2.1. *Let $u \in H^2(\Gamma)$ then for every $\eta > 0$ we have*

$$\|\partial u\|_{L^2(\Gamma)} \leq C \left(\eta^{1/2} \|u\|_{L^2(\Gamma)} + \eta^{-1/2} \|u\|_{H^2(\Gamma)} \right).$$

Proof. Let $u \in H^2(\Gamma)$. We first claim that for every $\alpha \in \mathcal{A}$ we have

$$\|\partial u_\alpha\|_{L^2([0,\ell_\alpha])} \leq \|u_\alpha\|_{L^2([0,\ell_\alpha])}^{1/2} \|\partial^2 u_\alpha\|_{L^2([0,\ell_\alpha])}^{1/2}. \tag{6.12}$$

Indeed consider first $v \in \mathscr{C}_c^\infty(\mathbb{R})$. By integration by parts we find that

$$\begin{aligned}\int_{\mathbb{R}} |\partial v|^2 \, dx &\leq \int_{\mathbb{R}} |v \partial^2 v| \, dx \\ &\leq \|v\|_{L^2(\mathbb{R})} \|\partial^2 v\|_{L^2(\mathbb{R})}\end{aligned}$$

where we used the Cauchy-Schwarz inequality to obtain the last line. This yields

$$\|\partial v\|_{L^2(\mathbb{R})} \leq \|v\|_{L^2(\mathbb{R})}^{1/2} \|\partial^2 v\|_{L^2(\mathbb{R})}^{1/2}.$$

We then obtain (6.12) by the density of $\mathscr{C}_c^\infty(\mathbb{R})$ in $H^2([0, \ell_\alpha])$ (see [Bre11, Theorem 8.7]). We can then apply Young's inequality to (6.12) to get

$$\|\partial u_\alpha\|_{L^2([0,\ell_\alpha])} \leq C \left(\eta^{1/2} \|u_\alpha\|_{L^2([0,\ell_\alpha])} + \eta^{-1/2} \|\partial^2 u_\alpha\|_{L^2([0,\ell_\alpha])} \right) \tag{6.13}$$

for every $\eta > 0$. Finally we can sum over $\alpha \in \mathcal{A}$ in (6.13) to obtain

$$\|\partial u\|_{L^2(\Gamma)} \leq C \left(\eta^{1/2} \|u\|_{L^2(\Gamma)} + \eta^{-1/2} \|\partial^2 u\|_{L^2(\Gamma)} \right).$$

□

We can now consider the linear operator $B = \partial$ with natural domain $D(B) = H^1(\Gamma)$. Using Theorem 3.2.9 and Proposition 3.2.6 we have $\|u\|_{H^2(\Gamma)} \leq C \|L_0 u\|_{L^2(\Gamma)}$ for every $u \in D(L)$ and from Theorem B.3.6 and Proposition 6.2.1 we deduce that for every $\alpha > \frac{1}{2}$ the fractional operator L_0^α (see Appendix B.3.3) satisfies $D(L_0^\alpha) \subset D(B)$. In particular there exists a continuous embedding $X_\alpha \hookrightarrow H^1(\Gamma)$, where X_α is $D(L_0^\alpha)$ endowed with the graph norm of L_0^α in $L^2(\Gamma)$.

Then as $f \in H^1(0, T, L^2(\Gamma))$ we deduce $f \in \mathscr{C}^{0,\sigma}([0, T], L^2(\Gamma))$ for $\sigma \in (0, 1/2]$. Also as $H_\alpha \in \mathscr{C}([0, \ell_\alpha], \mathbb{R})$ and $|\partial_p H_\alpha|$ is bounded we deduce that H_α is Lipschitz continuous for every $\alpha \in \mathcal{A}$. We therefore have for $(t_1, u_1), (t_2, u_2) \in (0, T) \times X_\alpha$

$$\begin{aligned}&\|f_0(t_1, u_1) - f_0(t_2, u_2)\|_{L^2(\Gamma)} \\ &\quad \leq \|f(t_1, \cdot) - f(t_2, \cdot)\|_{L^2(\Gamma)} + \lambda_0 \|u_1 - u_2\|_{L^2(\Gamma)} + \|H(\cdot, \partial u_1) - h(\cdot, \partial u_2)\|_{L^2(\Gamma)} \\ &\quad \leq K_1 |t_1 - t_2|^\sigma + K_2 \left(\|u_1 - u_2\|_{L^2(\Gamma} + \|\partial u_1 - \partial u_2\|_{L^2(\Gamma)} \right) \\ &\quad \leq L \left(|t_1 - t_2|^\sigma + \|u_1 - u_2\|_{H^1(\Gamma)} \right) \\ &\quad \leq K_3 \left(|t_1 - t_2|^\sigma + \|u_1 - u_2\|_{X_\alpha} \right).\end{aligned}$$

This proves that Assumption 7 holds for $U = (0, T) \times X_\alpha$ for $\alpha > \frac{1}{2}$. Moreover

$$\begin{aligned}\|f_0(t, u)\|_{L^2(\Gamma)} &\leq K \left(1 + \|u\|_{L^2(\Gamma)} + \|\partial u\|_{L^2(\Gamma)} \right) \\ &\leq K \left(1 + \|u\|_{H^1(\Gamma)} \right) \\ &\leq \tilde{K} \left(1 + \|u\|_{X_\alpha} \right)\end{aligned}$$

We can now apply Theorems B.3.7 and B.3.8[2] to obtain the following result.

[2] Reverting time.

Lemma 6.2.2. *Under Assumption 5 there exists a unique solution*

$$u \in \mathscr{C}([0,T], L^2(\Gamma)) \cap \mathscr{C}^1([0,T), H^1(\Gamma)) \cap \mathscr{C}([0,T), H^2(\Gamma))$$

of (6.10). *Moreover $\partial_t u$ is locally Hölder continuous in t with values in $H^1(\Gamma)$.*

The regularity of the time derivative then allows to directly deduce the following better result. The estimates are taken from [ADLT20, Theorem 4.5].

Theorem 6.2.3. *Under Assumption 5 there exists a unique classical solution*

$$u \in \mathscr{C}([0,T], L^2(\Gamma)) \cap \mathscr{C}^1([0,T), H^1(\Gamma)) \cap \mathscr{C}([0,T), H^3(\Gamma))$$

of (6.10) *with*

$$\|u\|_{L^2(0,T,H^3(\Gamma))} + \|\partial_t u\|_{L^2(0,T,H^1(\Gamma))} \leq C$$

where C depends only on u_T, μ, H and f. Moreover $t \mapsto \partial_t u(t,\cdot)$ is locally Hölder continuous with values in $H^1(\Gamma)$.

Appendix A

Proofs

This appendix contains the proofs of various results stated in the body of the text.

A.1 Properties of networks

Proof A.1.1 (Proof of Proposition 2.1.5)**.** Let $x, y \in \Gamma$ with $x \neq y$. First notice that $|x - y| \leq d(x, y)$ always holds. For the second estimate consider the quantity

$$C = \sup_{\substack{x,y\in\Gamma \\ x\neq y}} \frac{d(x,y)}{|x-y|}.$$

If x and y belong to the same edge we see that $d(x, y) = |x - y|$. Therefore we can consider a maximizing sequence $(x_n, y_n)_{n\in\mathbb{N}}$ such that there exists $\alpha, \beta \in \mathcal{A}$ with $\Gamma_\alpha \cap \Gamma_\beta = v \in \mathcal{V}$ and such that Γ_α and Γ_β are not parallel, otherwise the conclusion of the first case also holds. See Fig. A.1 for notations.

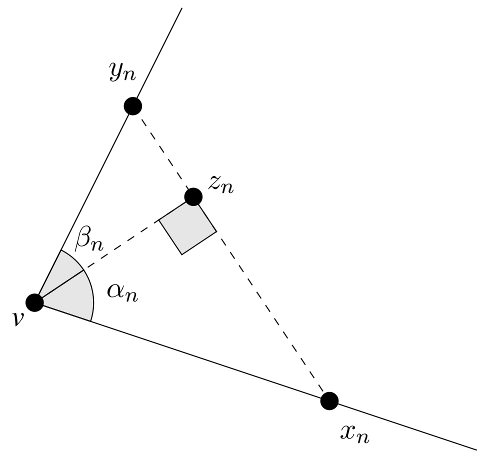

Figure A.1 – The maximizing sequence.

In the case where $\lim_{n\to\infty}\alpha_n = 0$ or $\lim_{n\to\infty}\beta_n = 0$ it is clear that $d(x_n, y_n)$ is close to $|x_n - y_n|$ and therefore C will be bounded. Hence we may assume, up to extracting a subsequence, that there exists some $\delta \in (0, \pi/2)$ such that $\frac{\pi}{2} \geq \alpha_n, \beta_n > \delta$ for every n. Then we will have

$$|x_n - y_n| = \sin(\alpha_n)|x_n - v| + \sin(\beta_n)|y_n - v| \geq \sin(\delta)d(x_n, y_n).$$

Therefore $C \leq \frac{1}{\sin(\delta)}$. This concludes the proof.

□

A.2 Properties of function spaces

Proof A.2.1 (Proof of Proposition 2.2.3)**.** Let $(u_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $(\mathscr{C}^k(\Gamma), \|\cdot\|_{\mathscr{C}^k(\Gamma)})$. Then for every $\epsilon > 0$ the exists $N_\epsilon \in \mathbb{N}$ such that for each $\alpha \in \mathcal{A}$ we have

$$\|u_{\alpha,n} - u_{\alpha,m}\|_{\mathscr{C}^k([0,\ell_\alpha])} = \sum_{0\leq j\leq k} \|\partial^j u_{\alpha,n} - \partial^j u_{\alpha,m}\|_{L^\infty(0,\ell_\alpha)} \leq \|u_n - u_m\|_{\mathscr{C}^k(\Gamma)} < \epsilon$$

for every $n, m \geq N_\epsilon$. This mean that $(u_{\alpha,n})_{n\in\mathbb{N}}$ is a Cauchy sequence in $C^k([0, \ell_\alpha])$ for each $\alpha \in \mathcal{A}$. We know that the spaces $\mathscr{C}^k([0, \ell_\alpha])$ are complete, hence each sequence $(u_{\alpha,n})_{n\in\mathbb{N}}$ converges to some $v_\alpha \in \mathscr{C}^k([0, \ell_\alpha])$. We now define $v_{|\Gamma_\alpha}(x) = v_\alpha(\pi_\alpha^{-1}(x))$ when $x \in \Gamma_\alpha$. To show that this function is well defined and belongs to $\mathscr{C}(\Gamma)$ notice that $(u_n)_{n\in\mathbb{N}}$ is also a Cauchy sequence in $\mathscr{C}(\Gamma)$, which we know to be complete, and hence must converge to some function which coincides with v on Γ_α for every $\alpha \in \mathcal{A}$. This implies that v must be continuous and that the value of $v(\pi_\alpha^{-1}(v_i))$ does not depend on α for every $i \in \mathcal{I}$. We have shown that $(u_n)_{n\in\mathbb{N}}$ converges to v in $(\mathscr{C}^k(\Gamma), \|\cdot\|_{\mathscr{C}^k(\Gamma)})$ which is then complete. □

Proof A.2.2 (Proof of Proposition 2.2.7)**.** We will need the following standard lemma.

Lemma A.2.1. *Let I be a bounded interval in $\mathbb{R}$ and $0 < \gamma < \theta \leq 1$. There is a compact embedding from $\mathscr{C}^{0,\theta}(I)$ to $\mathscr{C}^{0,\gamma}(I)$ and for $u \in \mathscr{C}^{0,\theta}(I)$ we have*

$$\|u\|_{\mathscr{C}^{0,\gamma}(I)} \leq C\|u\|_{\mathscr{C}^{0,\theta}(I)}.$$

Proof. Let $u \in \mathscr{C}^{0,\theta}(I)$ and $x, y \in I$ then from the boundedness of I we have

$$\frac{|u(x) - u(y)|}{|x - y|^\gamma} = \frac{|x - y|^{\theta-\gamma}}{|x - y|^{\theta-\gamma}} \frac{|u(x) - u(y)|}{|x - y|^\gamma} \leq \operatorname{diam}(I)^{\theta-\gamma} \frac{|u(x) - u(y)|}{|x - y|^\theta}$$

which implies that

$$\|u\|_{\mathscr{C}^{0,\gamma}(I)} \leq C\|u\|_{\mathscr{C}^{0,\theta}(I)}.$$

The compactness of the embedding the follows from Ascoli's theorem ([Dix84, Theorem 6.3.1]).

□

Let us now turn to the proof of Proposition 2.2.7. Let $u_n \in \mathscr{C}^{0,\theta}(\Gamma)$ be a bounded sequence and $0 < \gamma < \theta$. Then $u_{n,\alpha} \in \mathscr{C}^{0,\theta}([0,\ell_\alpha])$ for every $\alpha \in \mathcal{A}$. By the compact embedding between Hölder spaces obtained in Lemma A.2.1 we have $u_{n,\alpha} \in \mathscr{C}^{0,\gamma}([0,\ell_\alpha])$ and $\|u_{n,\alpha}\|_{\mathscr{C}^{0,\gamma}([0,\ell_\alpha])} \leq C_\alpha \|u_{n,\alpha}\|_{\mathscr{C}^{0,\theta}([0,\ell_\alpha])}$ and there exists a $u_\alpha \in \mathscr{C}^{0,\gamma}([0,\ell_\alpha])$ and a subsequence which we still denote $(u_{n,\alpha})_{n\in\mathbb{N}}$ which converges to u_α in $\mathscr{C}^{0,\gamma}([0,\ell_\alpha])$. This defines a function $u \in PC(\Gamma)$ such that $u_\alpha \in \mathscr{C}^{0,\gamma}([0,\ell_\alpha])$. To check that u is indeed continuous at the junctions consider any vertex $v_i \in \mathcal{V}$ and $\alpha, \beta \in \mathcal{A}_i$. Then

$$\begin{aligned} |u_{|\Gamma_\alpha}(v_i) - u_{|\Gamma_\beta}(v_i)| &\leq |u_{|\Gamma_\alpha}(v_i) - u_{n|\Gamma_\alpha}(v_i)| + |u_{|\Gamma_\beta}(v_i) - u_{n|\Gamma_\alpha}(v_i)| \\ &= |u_{|\Gamma_\alpha}(v_i) - u_{n|\Gamma_\alpha}(v_i)| + |u_{|\Gamma_\beta}(v_i) - u_{n|\Gamma_\beta}(v_i)| \end{aligned}$$

where we used the continuity of u_n at the junction and the last term converges to 0 as n tends to infinity. Therefore the sequence u_n convergences to u in $\mathscr{C}^{0,\gamma}(\Gamma)$ and the result is proved. $\square$

Proof A.2.3 (First proof of Proposition 2.2.12)**.** Let $(u_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $W^{k,p}(\Gamma)$. Then, as $\|v_\alpha\|_{W^{k,p}(0,\ell_\alpha)} \leq \|v\|_{W^{k,p}(\Gamma)}$, for each $\alpha \in \mathcal{A}$ the sequence $(u_{\alpha,n})_{n\in\mathbb{N}}$ is a Cauchy sequence in $W^{k,p}(0,\ell_\alpha)$. This space being a Banach space, there exist functions $v_\alpha \in W^{k,p}(0,\ell_\alpha)$ which are the limits of $u_{\alpha,n}$ as n tends to infinity in $W^{k,p}(0,\ell_\alpha)$. From the standard Sobolev inequality [Bre11, Theorem 8.8] we have $u_{n,\alpha}, v_\alpha \in \mathscr{C}([0,\ell_\alpha])$ (notice that this implies $u_{n,|\Gamma_\alpha}, v_{|\Gamma_\alpha} \in \mathscr{C}(\Gamma_\alpha)$) and

$$\|v_{|\Gamma_\alpha} - u_{n|\Gamma_\alpha}\|_{\mathscr{C}(\Gamma_\alpha)} = \|v_\alpha - u_{n,\alpha}\|_{\mathscr{C}([0,\ell_\alpha])} \leq C\|v_\alpha - u_{n,\alpha}\|_{W^{k,p}(0,\ell_\alpha)}.$$

and taking the limit as n tends to infinity shows that $u_{n,|\Gamma_\alpha}$ converges uniformly to $v_{|\Gamma_\alpha}$ on Γ_α. Then notice that for $\alpha, \beta \in \mathcal{A}_i$,

$$\sup_{x\in\Gamma_\alpha\cup\Gamma_\beta} |v_{|\Gamma_\alpha\cup\Gamma_\beta}(x) - u_{n|\Gamma_\alpha\cup\Gamma_\beta}(x)| = \max\left\{\|v_{|\Gamma_\alpha} - u_{n|\Gamma_\alpha}\|_{\mathscr{C}(\Gamma_\alpha)}, \|v_{|\Gamma_\beta} - u_{n|\Gamma_\beta}\|_{\mathscr{C}(\Gamma_\beta)}\right\}$$

and taking the limit as n tends to infinity we find that we have in fact uniform convergence on $\Gamma_\alpha \cup \Gamma_\beta$. In particular we have

$$v_{|\Gamma_\alpha}(v_i) = \lim_{\substack{x\to v_i \\ x\in\Gamma_\alpha}} v_{|\Gamma_\alpha}(x) = \lim_{\substack{x\to v_i \\ x\in\Gamma_\beta}} v_{|\Gamma_\beta}(x) = v_{|\Gamma_\beta}(v_i)$$

for every $i \in \mathcal{I}$ and $\alpha, \beta \in \mathcal{A}_i$. This shows that $v \in \mathscr{C}(\Gamma)$. We have proved that u_n converges to v in $W^{k,p}(\Gamma)$ which is thus a Banach space.

For $H^k(\Gamma)$, consider the bilinear form

$$H^k(\Gamma) \times H^k(\Gamma) \ni (u,v) \mapsto \sum_{j=0}^{k} \int_\Gamma \partial^k u(x) \partial^k v(x)\, dx \in \mathbb{R}.$$

One can easily see that this mapping defines a scalar product on $H^k(\Gamma)$ whose associated norm is the norm defined in Definition 2.2.10. $\square$

Proof A.2.4 (Second proof of Proposition 2.2.12)**.** We begin with the following lemma.

Lemma A.2.2. *Let $a < b < c$ be real numbers and $u : [a,c] \to \mathbb{R}$ be a function such that $u_{|[a,b]} \in W^{1,p}(a,b)$, $u_{|[b,c]} \in W^{1,p}(b,c)$ and u is continuous at b. Then $u \in W^{1,p}(a,c)$.*

Proof. First we clearly have $\|u\|_{L^p(a,c)} < \infty$. Now let us compute the distributional derivative of u on (a,c). For this consider $\phi \in \mathscr{C}_c^\infty(a,c)$. As we assume $u(b^-) = u(b^+)$ we have

$$\begin{aligned}\int_a^c u(x)\partial\phi(x)\,dx &= \int_a^b u_{|(a,b)}(x)\partial\phi(x)\,dx + \int_b^c u_{|(b,c)}(x)\partial\phi(x)\,dx \\ &= \phi(b)\left(u(b^-) - u(b^+)\right) - \int_a^b \partial u_{|(a,b)}(x)\phi(x)\,dx - \int_b^c \partial u_{|(b,c)}(x)\phi(x)\,dx \\ &= -\int_a^b \partial u_{|(a,b)}(x)\phi(x)\,dx - \int_b^c \partial u_{|(b,c)}(x)\phi(x)\,dx.\end{aligned}$$

Therefore we can define

$$\partial u(x) = \begin{cases} \partial u_{|(a,b)}(x) & \text{if } x \in (a,b) \\ \partial u_{|(b,c)}(x) & \text{if } x \in (b,c), \end{cases}$$

and we have obtained

$$\int_a^c u(x)\partial\phi(x)\,dx = -\int_a^c \partial u(x)\phi(x)\,dx.$$

Which means that ∂u is indeed the distributional derivative of u on (a,c)[1]. From the assumptions on u, we also have $\|\partial u\|_{L^p(a,c)} < \infty$, which concludes the proof. $\square$

Define the functions v_α for each $\alpha \in \mathcal{A}$ and v just as in the first proof. Recall that v satisfies $v_{|\Gamma_\alpha} \in \mathscr{C}(\Gamma_\alpha)$ for every $\alpha \in \mathcal{A}$. Choose $i \in \mathcal{I}$ and $\alpha, \beta \in \mathcal{A}_i$. We assume that v_i is such that $\mathsf{v}_i = \pi_\alpha(0) = \pi_\beta(0)$ [2]. We can parameterize $\Gamma_\alpha \cup \Gamma_\beta$ by the following continuous function

$$\pi_{\alpha,\beta} : \begin{array}{rcl} [0, \ell_\alpha + \ell_\beta] & \to & \Gamma \\ y & \mapsto & \begin{cases} \pi_\alpha(\ell_\alpha - y) & \text{if } y \in [0, \ell_\alpha], \\ \pi_\beta(y - \ell_\alpha) & \text{if } y \in [\ell_\alpha, \ell_\alpha + \ell_\beta], \end{cases} \end{array}$$

Notice that $\mathsf{v}_i = \pi_{\alpha,\beta}(\ell_\alpha)$. The functions u_n being continuous on $\Gamma_\alpha \cup \Gamma_\beta$, the function

$$u_n^{\alpha,\beta} : \begin{array}{rcl} [0, \ell_\alpha + \ell_\beta] & \to & \mathbb{R} \\ y & \mapsto & u_n \circ \pi_{\alpha,\beta}(y) \end{array}$$

are well defined and continuous on $[0, \ell_\alpha + \ell_\beta]$. Moreover $(u_n^{\alpha,\beta})_{n\in\mathbb{N}}$ is a Cauchy sequence in $W^{1,p}(0, \ell_\alpha + \ell_\beta)$ according to Lemma A.2.2 and they must thus converge to some $w^{\alpha,\beta} \in$

[1] Continuity of u at b is a necessary assumptions for this to hold. Otherwise we need to add a Dirac at b and the distributional derivative is not a function anymore.

[2] This is always possible as explained in Remark 2.1.1.

$W^{1,p}(0, \ell_\alpha + \ell_\beta)$. By standard Sobolev inequalities we also have $w^{\alpha,\beta} \in \mathscr{C}([0, \ell_\alpha + \ell_\beta])$. Then we can define the function

$$w : \begin{array}{rcl} \Gamma_\alpha \cup \Gamma_\beta & \to & \mathbb{R} \\ x & \mapsto & w^{\alpha,\beta} \circ \pi_{\alpha,\beta}^{-1}(x) \end{array}$$

which is then a continuous function on $\Gamma_\alpha \cup \Gamma_\beta$. Moreover one also has

$$w^{\alpha,\beta}(y) = v_\alpha(\ell_\alpha - y) \quad \forall y \in [0, \ell_\alpha],$$
$$w^{\alpha,\beta}(\ell_\alpha + y) = v_\beta(y) \quad \forall y \in [0, \ell_\beta].$$

This gives us

$$\lim_{\substack{x \to v_i \\ x \in \Gamma_\alpha}} v_{|\Gamma_\alpha}(x) = \lim_{\substack{x \to v_i \\ x \in \Gamma_\alpha}} w(x) = \lim_{\substack{x \to v_i \\ x \in \Gamma_\beta}} w(x) = \lim_{\substack{x \to v_i \\ x \in \Gamma_\beta}} v_{|\Gamma_\beta}(x)$$

which shows that $v \in \mathscr{C}(\Gamma_\alpha \cup \Gamma_\beta)$. As we chose $v_i \in \mathcal{V}$, and $\alpha, \beta \in \mathcal{A}_i$ arbitrarily, we have shown that $v \in \mathscr{C}(\Gamma)$ and the result follows.

□

Proof A.2.5 (Proof of Proposition 2.2.13)**.** The continuous injection follows from the continuous embedding $\mathscr{C}(\Gamma) \hookrightarrow L^q(\Gamma)$. Assume now that $1 \le q < \infty$ and let $(u_n)_{n\in\mathbb{N}}$ be a bounded sequence in $W^{1,1}(\Gamma)$. Then each sequence $(u_{\alpha,n})_{n\in\mathbb{N}}$ is a bounded sequence in $W^{1,1}(0, \ell_\alpha)$. Using once again [Bre11, Theorem 8.8] there exists a subsequence that we still denote $(u_{\alpha,n})_{n\in\mathbb{N}}$ which converges to some u_α in $L^q(0, \ell_\alpha)$. Now we define $u(x) = u_\alpha \circ \pi_\alpha^{-1}(x)$ for $x \in \Gamma_\alpha$ and clearly u_n converges to u in $L^q(\Gamma)$. This show that the embedding is compact. □

Proof A.2.6 (Proof of Proposition 2.2.15)**.** Let $u \in W^{1,p}(\Gamma)$, then by definition $u \in \mathscr{C}(\Gamma)$ and $\|u\|_{\mathscr{C}(\Gamma)} \le C_1 \|u\|_{W^{1,p}(\Gamma)}$ by Proposition 2.2.13. We then also have $u_\alpha \in W^{1,p}(0, \ell_\alpha)$ for every $\alpha \in \mathcal{A}$ which by the standard Morrey inequality (see [Eva10, Theorem 4 p.282]) means that $u_\alpha \in \mathscr{C}^{0,\theta}([0, \ell_\alpha])$ with $0 < \theta \le 1 - \frac{1}{p}$. Moreover one has $\|u_\alpha\|_{\mathscr{C}^{0,\theta}([0,\ell_\alpha])} \le C_\alpha \|u_\alpha\|_{W^{1,p}(0,\ell_\alpha)}$. Hence

$$\begin{aligned} \|u\|_{\mathscr{C}^{0,\theta}(\Gamma)} &= \|u\|_{\mathscr{C}(\Gamma)} + \max_{\alpha\in\mathcal{A}} \|u_\alpha\|_{\mathscr{C}^{0,\theta}([0,\ell_\alpha])} \le C_1 \|u\|_{W^{1,p}(\Gamma)} + C_2 \max_{\alpha\in\mathcal{A}} \|u_\alpha\|_{W^{1,p}(0,\ell_\alpha)} \\ &\le C \|u\|_{W^{1,p}(\Gamma)}. \end{aligned}$$

Thus $u \in \mathscr{C}^{0,\theta}(\Gamma)$. The compact embedding the follows from there is a compact injection from $\mathscr{C}^{0,\theta}(\Gamma)$ into $\mathscr{C}^{0,\gamma}(\Gamma)$ for $0 < \gamma < \theta$ in Proposition 2.2.7. □

A.3 Linear elliptic equations

Proof A.3.1 (Proof of Theorem 3.1.1)**.** We only prove the first point the second one can be proved in a very similar way.

First notice that the assumptions imply $u_\alpha \in \mathscr{C}^2(0, \ell_\alpha) \cap \mathscr{C}([0, \ell_\alpha])$, $c_\alpha = 0$ on $(0, \ell_\alpha)$ and $L_\alpha u_\alpha \le 0$ on $(0, \ell_\alpha)$. Hence that assumptions of the weak maximum principle ([Eva10, Theorem 1 p.346], [GT01, Theorem 3.1]) are fulfilled and we know that

$$\max_{y\in(0,\ell_\alpha)} u_\alpha(y) = \max\{u_\alpha(0), u_\alpha(\ell_\alpha)\}.$$

This is true for every $\alpha \in \mathcal{A}$ and as $\max_{x\in\Gamma} u(x) = \max_{\alpha\in\mathcal{A}} \max_{y\in(0,\ell_\alpha)} u_\alpha(y)$ and $u \in \mathscr{C}(\Gamma)$ we obtain

$$\max_{x\in\Gamma} u(x) = \max_{x\in\mathcal{V}} u(x).$$

□

Proof A.3.2 (Proof of Lemma 3.2.5)**.** Without loss of generality we may assume $\theta \leq \gamma$. Let $x, y \in I$, we have

$$\begin{aligned}|(u(x)+v(x)) - (u(y)+v(y))| &\leq |u(x)-u(y)| + |v(x)-v(y)| \\ &\leq C_1|x-y|^\theta + C_2|x-y|^\gamma \\ &\leq C_1|x-y|^\theta + C_2|I|^{\gamma-\theta}|x-y|^\theta \\ &\leq C|x-y|^\theta\end{aligned}$$

for some positive constant C. This shows that $(u+v) \in \mathscr{C}^{0,\theta}(I)$. Also

$$\begin{aligned}|u(x)v(x) - u(y)v(y)| &= |u(x)(v(x)-v(y)) - v(y)(u(y)-u(x)| \\ &\leq |u(x)||v(x)-v(y)| + |v(y)||u(x)-u(y)| \\ &\leq K_1|x-y|^\gamma + K_2|x-y|^\theta \\ &\leq K|x-y|^\theta,\end{aligned}$$

which proves that $uv \in \mathscr{C}^{0,\theta}(I)$.

Finally if $u \geq k > 0$ for some constant k we have

$$\begin{aligned}\left|\frac{1}{u(x)} - \frac{1}{u(y)}\right| = \left|\frac{u(y)-u(x)}{u(x)u(y)}\right| &\leq \frac{|u(x)-u(y)|}{k^2} \\ &\leq \frac{|x-y|^\theta}{k^2}\end{aligned}$$

which proves that $\frac{1}{u} \in \mathscr{C}^{0,\theta}(I)$. □

Proof A.3.3 (Proof of Theorem 3.2.9)**.** We are going to use a Fredholm alternative argument. First we know thanks to Theorem 3.2.3 that for λ_0 large enough and any $f \in H^{-1}(\Gamma)$, the problem

$$\text{(A.1)} \qquad \left|\begin{array}{l}\text{find } u \in H^1(\Gamma) \text{ such that} \\ B(u,v) + \lambda_0(u,v)_{L^2(\Gamma;\psi)} = \langle f, v\rangle_{H^{-1},H^1} \quad \text{for every } v \in H^1(\Gamma)\end{array}\right.$$

has a unique solution $u \in H^1(\Gamma)$ and $\|u\|_{H^1(\Gamma} \leq C\|f\|_{H^{-1}}$. In what follows we denote

$$B_{\lambda_0}(u,v) = B(u,v) + \lambda_0(u,v)_{L^2(\Gamma;\psi)}.$$

Then for λ_0 large enough we can define the continuous linear operator $A_{\lambda_0} : H^{-1}(\Gamma) \to H^1(\Gamma)$ mapping $f \in H^{-1}(\Gamma)$ to the solution $u \in H^1(\Gamma)$ of (A.1) and we have $\|A_{\lambda_0}\|_{\mathcal{L}(H^{-1}(\Gamma),H^1(\Gamma))} \leq \frac{1}{C}$. Notice that $u \in H^1(\Gamma)$ solves ($\mathscr{E}'$) if, and only if, it verifies

$$B_{\lambda_0}(u,v) = \lambda_0(u,v)_{L^2(\Gamma;\psi)} + \langle f, v\rangle_{H^{-1},H^1} \quad \forall v \in H^1(\Gamma),$$

that is if and only if $u = A_{\lambda_0}(\lambda_0 u + f)$ or equivalently

$$(I - \lambda_0 A_{\lambda_0})u = A_{\lambda_0} f. \tag{A.2}$$

Suppose $u \in H^{-1}(\Gamma)$ is such that $(I - \lambda_0 A_{\lambda_0})u = 0$, then $u = \lambda_0 A_{\lambda_0} u \in H^1(\Gamma)$ and

$$B_{\lambda_0}(u/\lambda_0, v) = (u, v)_{L^2(\Gamma;\psi)} \quad \forall v \in H^1(\Gamma),$$

which is equivalent to

$$B(u, v) = 0 \quad \forall v \in H^1(\Gamma).$$

According to Lemma 3.2.8 this implies that $u = 0$ and hence $N(I - \lambda_0 A_{\lambda_0}) = \{0\}$.

Since the injection $H^1(\Gamma) \hookrightarrow L^2(\Gamma) \hookrightarrow H^{-1}(\Gamma)$ is compact according to Proposition 2.2.13, we can extend A_{λ_0} into the operator $\tilde{A}_{\lambda_0} : H^{-1}(\Gamma) \to H^{-1}(\Gamma)$ which is compact (see [Bre11, Proposition 6.3]). Denote J the canonical compact operator associated with this injection, we thus have $\tilde{A}_{\lambda_0} = J \circ A_{\lambda_0}$ and as $Jf = f$ for every $f \in H^{-1}(\Gamma)$ we also have $I - \lambda_0 A_{\lambda_0} = J \circ (I - \lambda_0 A_{\lambda_0})$. Moreover as $N(J) = \{0\}$ we clearly have $N(I - \lambda_0 \tilde{A}_{\lambda_0}) = N(I - \lambda_0 A_{\lambda_0}) = \{0\}$ and the Fredholm alternative (see [Bre11, Theorem 6.6]) then states that $I - \lambda_0 \tilde{A}_{\lambda_0}$ is a bijective bounded linear operator from $H^{-1}(\Gamma)$ into itself. In particular we can define the continuous linear operator $(I - \lambda_0 \tilde{A}_{\lambda_0})^{-1} : H^{-1}(\Gamma) \to H^{-1}(\Gamma)$. Finally note if $g \in H^1(\Gamma)$ then $H^{-1}(\Gamma) \ni w = (I - \lambda_0 \tilde{A}_{\lambda_0})^{-1} g$ if, and only if, $(I - \lambda_0 \tilde{A}_{\lambda_0})w = g$ which is equivalent to $w = g + \lambda_0 \tilde{A}_{\lambda_0} w$. As $\tilde{A}_{\lambda_0} w \in H^{-1}(\Gamma) \cap H^1(\Gamma)$ we must have $w \in H^1(\Gamma)$ and[3]

$$\|w\|_{H^1(\Gamma)} \le \|g\|_{H^1(\Gamma)} + \lambda_0 \|A_{\lambda_0} w\|_{H^1(\Gamma)} \le \|g\|_{H^1(\Gamma)} + \lambda_0 K \|w\|_{W^\star} \tag{A.3}$$

Coming back to (A.2) we see that

$$u = (I - \lambda_0 \tilde{A}_{\lambda_0})^{-1} A_{\lambda_0} f$$

is the unique weak solution of ($\mathscr{E}'$) and the continuity of A_{λ_0} and $(I - \lambda_0 \tilde{A}_{\lambda_0})^{-1}$ gives $\|u\|_{H^{-1}} \le C\|f\|_{H^{-1}}$ and from (A.3) we get

$$\|u\|_{H^1(\Gamma)} \le C' \|f\|_{H^{-1}}.$$

□

Proof A.3.4 (Proof of Proposition 3.2.10)**.** We have $Lu = f \ge 0$, therefore the weak maximum principle 3.1.2 states that

$$\min_{x\in\Gamma} u(x) \ge \min_{x\in\mathcal{V}} u^-(x)$$

If $\min_{x\in\mathcal{V}} u^-(x) = 0$, the result is proved. Let us thus suppose that $u^-(v_i) < 0$ is minimal for u for some $i \in \mathcal{I}$. From the continuity of u there exists in a neighborhood of v_i such that $u^- = u$ on this neighborhood. Hence u^- also satisfies the Kirchhoff condition at v_i and Proposition 2.2.4 tells us that $\partial_\alpha u^-(v_i) = 0$ for every $\alpha \in \mathcal{A}_i$. From the continuity of the coefficients and the fact that $u \in \mathscr{C}^2(\Gamma)$ we have (assuming $v_i = \pi_\alpha(0)$)

$$-a_\alpha(0)\partial^2 u_\alpha^-(0) + c_\alpha(0) u_\alpha^-(0) = f_\alpha(0) \ge 0.$$

[3] As $w \in H^{-1}(\Gamma) \cap H^1(\Gamma)$ we can identify w with $J^{-1}w$ where J^{-1} is defined form $J(H^1(\Gamma)) \subset H^{-1}(\Gamma) \cap H^1(\Gamma)$ to $H^1(\Gamma)$.

As we assume $c_\alpha(0) > 0$, and $u^-(v_i) < 0$ we have

$$\partial^2 u_\alpha^-(0) \leq \frac{c_\alpha(0) u_\alpha^-(0)}{a_\alpha(0)} < 0.$$

This implies that u_α^- is strictly concave near 0 which contradicts the fact that v_i is a minimum point. Hence we must have $\min_{x\in\Gamma} u(x) \geq 0$. $\square$

Proof A.3.5 (Proof of Proposition 3.3.1)**.** We write $\overline{\partial\phi} = \|\partial\phi\|_{L^\infty(\Gamma)}$ and $\underline{\phi} = \min_{x\in\Gamma}\phi(x) > 0$. Recall that $B_\lambda(\cdot,\cdot) = B(\cdot,\cdot) + \lambda(\cdot,\cdot)_{L^2(\Gamma;\phi)}$. We have

$$\begin{aligned}
B_\lambda(w,w) &= \int_\Gamma a\partial w \partial(w\phi) - bw\partial(w\phi) + \lambda|w|^2\phi \, dx \\
&= \int_\Gamma a|\partial w|^2\phi + (a\partial\phi - b\phi)w\partial w - b|w|^2\partial\phi + \lambda|w|^2\phi \, dx \\
&\geq \int_\Gamma \omega|\partial w|^2\phi - \left(\|a\|_{L^\infty(\Gamma)}\frac{\overline{\partial\phi}}{\underline{\phi}} + \|b\|_{L^\infty(\Gamma)}\right)|w\partial w|\phi + \left(\lambda - \|b\|_{L^\infty}\frac{\overline{\partial\phi}}{\underline{\phi}}\right)|w|^2\phi \, dx.
\end{aligned}$$

Therefore, if we denote $K = \left(\|a\|_{L^\infty(\Gamma)}\frac{\overline{\partial\phi}}{\underline{\phi}} + \|b\|_{L^\infty(\Gamma)}\right)$ we have

$$B_\lambda(w,w) \geq \omega\|\partial_w\|^2_{L^2(\Gamma,\phi)} + \left(\lambda - \|b\|_{L^\infty(\Gamma)}\frac{\overline{\partial\phi}}{\underline{\phi}}\right)\|w\|^2_{L^2(\Gamma;\phi)} - K\int_\Gamma |w\partial w|\phi \, dx.$$

And using Young's inequality we find

$$B_\lambda(w,w) \geq \left(\omega - \frac{\epsilon K}{2}\right)\|\partial_w\|^2_{L^2(\Gamma,\phi)} + \left(\lambda - \|b\|_{L^\infty(\Gamma)}\frac{\overline{\partial\phi}}{\underline{\phi}} - \frac{K}{2\epsilon}\right)\|\partial w\|^2_{L^2(\Gamma;\phi)}$$

for every $\epsilon > 0$. Hence we can choose $\epsilon > 0$ small enough so that $\omega - \frac{\epsilon K}{2} > 0$ and then choose $\lambda = \lambda_0$ large enough to have $\lambda - \|b\|_{L^\infty(\Gamma)}\frac{\overline{\partial\phi}}{\underline{\phi}} - \frac{K}{2\epsilon} > 0$. Once we have done this there exists a positive constant C such that

$$B_{\lambda_0}(w,w) \geq C(\|w\|^2_{L^2(\Gamma;\phi)} + \|\partial w\|^2_{L^2(\Gamma;\phi)}) = C\|w\|^2_{W_\phi}.$$

To conclude the proof it suffices to apply the Lax-Milgram theorem.

$\square$

Proof A.3.6 (Proof of Theorem 3.3.2)**.** The argument is taken from [ADLT19, Theorem 2.7].
Existence : According to Proposition 3.3.1, for λ_0 large enough, there exists a unique weak solution $\hat{w} \in W$ to the following problem

$$\text{(A.4)} \qquad \begin{cases} -\partial(a\partial w) + \partial(bw) + \lambda_0 w = \lambda_0 h, & \text{on } \Gamma\setminus\mathcal{V}, \\ \frac{w_{|\Gamma_\alpha}(v_i)}{\gamma_{i,\alpha}} = \frac{w_{|\Gamma_\beta}(v_i)}{\gamma_{i,\beta}}, & \forall\alpha,\beta\in\mathcal{A}_i,\ \forall i\in\mathcal{I}, \\ \sum_{\alpha\in\mathcal{A}_i} a_{|\Gamma_\alpha}\partial_\alpha w_{|\Gamma_\alpha}(v_i) - n_{i,\alpha}w_{|\Gamma_\alpha}(v_i)a_{|\Gamma_\alpha}(v_i) = 0 & \forall i\in\mathcal{I}, \end{cases}$$

for $h \in L^2(\Gamma)$ with $\|\hat{w}\|_W \leq \lambda_0 C \|h\|_{L^2(\Gamma)}$. Hence we may define a continuous linear operator $T : L^2(\Gamma) \to W$ mapping h to $\hat{w}$. Now $w \in W$ is a weak solution of (3.6) if, and only if, it is a fixed point of T. Therefore our goal here is to apply Schauder's fixed point theorem.

We consider the following subset of $L^2(\Gamma)$

$$K = \left\{ w \in L^2(\Gamma) : w \geq 0 \text{ and } \int_\Gamma w \, dx = 1 \right\}.$$

We claim that $T(K) \subset K$. Indeed consider first the constant test-function $v = 1$. Then if we multiply by v in the equation and integrate by parts we find

$$\int_\Gamma -\partial(a\partial w)v + \partial(bw)v + \lambda_0 wv \, dx = \lambda_0 \int_\Gamma wv \, dx = \lambda_0 \int_\Gamma hv \, dx$$

and we deduce that

$$\int_\Gamma w \, dx = \int_\Gamma h \, dx.$$

Moreover let now $w \in W$ be a weak solution of (A.4) and consider w^- its negative part. Notice that w^- also belongs to W. Then as $\partial w \partial w^- = |\partial w^-|^2$ and $ww^- = |w^-|^2$ we have

$$\begin{aligned} B_{\lambda_0}(w, w^-) &= \int_\Gamma a|\partial w^-|^2\phi + (a\partial\phi - b\phi)w^-\partial w^- - b|w^-|^2\partial\phi + \lambda_0|w^-|^2\phi \, dx \\ &= \lambda_0 \int_\Gamma hw^- \, dx. \end{aligned}$$

Notice that if h is nonnegative, then the last term is nonpositive while we can assume, up to choosing an even larger λ_0 by the same argument as in Proof A.3.5, that the left-hand side is nonnegative. Therefore we must have $w^- = 0$ which implies that w is nonnegative. This proves our claim.

We now prove that K is precompact in $L^2(\Gamma)$. Indeed let $h \in K$ and $w = T(h) \in K \cap W$. Then from the coercivity of the bilinear form we have

$$C\|w\|_W^2 \leq B_{\lambda_0}(w, w) = \int_\Gamma hw\phi \, dx.$$

As $w\phi \in H^1(\Gamma)$ and because we have the continuous injection $H^1(\Gamma) \hookrightarrow L^\infty(\Gamma)$ we deduce that there exists a positive constant M such that

$$\int_\Gamma hw\phi \, dx \leq M\|w\phi\|_{L^\infty(\Gamma)} \int_\Gamma h \, dx = M\|w\|_{L^\infty}\bar{\phi} \leq M\bar{\phi}\|w\|_W.$$

Hence we have

$$\|w\|_W \leq \frac{M\bar{\phi}}{C}$$

which shows that $T(K)$ is bounded in W. Now we know that there is a compact injection $H^1(\Gamma) \hookrightarrow L^2(\Gamma)$ and a continuous injection $W \hookrightarrow H^1(\Gamma)$, given by $W \ni w \mapsto w\phi$. Therefore there exists a compact embedding $W \hookrightarrow L^2(\Gamma)$. We can thus extend T to a compact linear operator with values in $L^2(\Gamma)$ and $T(K)$ is then precompact in $L^2(\Gamma)$.

Finally, noticing that K is convex, we are able to apply Schauder's fixed point theorem (see [GT01, Corollary 11.2]) to claim that T admits at least one fixed point. Which implies the existence of a nonnegative solution of (3.6) satisfying $\int_\Gamma w\,dx = 1$ following to remark at the beginning of the proof.

Uniqueness : We are going to apply Fredholm's alternative. We consider the following problem

$$\begin{cases} -\partial(a\partial u) - b\partial u + \lambda_0 u = \lambda_0 g \quad \text{on } \Gamma \setminus \mathcal{V}, \\ u_{|\Gamma_\alpha}(v_i) = u_{|\Gamma_\beta}(v_i) \quad \forall \alpha, \beta \in \mathcal{A}_i,\ i \in \mathcal{I}, \\ \sum_{\alpha \in \mathcal{A}_i} \gamma_{i,\alpha} a_{|\Gamma_\alpha}(v_i) \partial_\alpha u(v_i) \quad \forall i \in \mathcal{I}. \end{cases} \tag{A.5}$$

According to Theorem 3.2.3, for λ_0 large enough, the bilinear form A associated to (A.5) (see (3.4)) is coercive and the problem has a unique weak solution $u \in H^1(\Gamma)$ such that $\|u\|_{H^1(\Gamma)} \leq \|g\|_{L^2(\Gamma)}$. Therefore we can define a continuous linear operator $S : L^2(\Gamma) \to H^1(\Gamma)$ which can be extended to a compact linear operator $S : L^2(\Gamma) \to L^2(\Gamma)$ such that $u = \lambda_0 S g$. Now notice that for $u \in H^1(\Gamma)$ and $w \in W$ we have $u\psi \in W$, $w\phi \in H^1(\Gamma)$ and

$$A(u, w\phi) = B(w, u\psi)$$

where B is the bilinear form defined in (3.9). Then using the fact that $\phi\psi = 1$ we obtain

$$\begin{aligned}(Tf, \lambda_0 g)_{L^2(\Gamma)} = (\phi T f, \lambda_0 g)_{L^2(\Gamma;\psi)} &= A(\lambda_0 S g, \phi T f) \\ &= B(\lambda_0 T f, \psi S g) = (\lambda_0 T f, \psi S g)_{L^2(\Gamma;\phi)} = (\lambda_0 f, S g)_{L^2(\Gamma)}.\end{aligned}$$

Therefore $S = T^\star$ in $L^2(\Gamma)$. From the Fredholm alternative (see [Bre11, Theorem 6.6]) we have that $\dim\ker(I - T) = \dim\ker(I - T^\star)$. And as $\ker(I - T^\star)$ only contains the constant constant functions (see Lemma 3.2.8), hence $\dim\ker(I - T^\star) = 1$. Therefore

$$\ker(I - \lambda_0 T) = \{w \in W : w = \mu w_0,\ w_0 \neq 0 \in W, \mu \in \mathbb{R}\}$$

and there only exists one such function such that $\int_\Gamma w\,dx = 1$. Finally noticing that any solution of (3.6) belongs to $\ker(I - T)$ this proves the uniqueness of the solution.

□

A.4 Linear parabolic equations

Proof A.4.1 (Proof of Lemma 4.1.2). Consider the following subspace of $L^2(\Gamma; \psi)$

$$E = \{u \in \mathscr{C}^\infty(\Gamma) :\ u_\alpha \in \mathscr{C}_c^\infty((0, \ell_\alpha)),\ \forall \alpha \in \mathcal{A}\}.$$

From the fact that $\mathscr{C}_c^\infty((0, \ell_\alpha))$ is dense in $L^2(0, \ell_\alpha))$ it is clear that E is dense in $L^2(\Gamma; \psi)$. Moreover for every $u \in E$, we have $u \in H^2(\Gamma; \psi)$ and because every u_α is compactly supported inside $(0, \ell_\alpha)$ we see that the function u must cancel on a neighborhood of every vertex. Therefore u must satisfy the Kirchoff condition. This proves that $E \subset D(L)$ and we conclude that $D(L)$ is dense in $L^2(\Gamma; \psi)$.

□

Proof A.4.2 (Proof of Lemma 4.1.3)**.** We first consider the operator $L_{\lambda_0} = L + \lambda_0 I$. Notice that $D(L_{\lambda_0}) = D(L)$. Let $(u_k)_{k\in\mathbb{N}}$ be a sequence in $D(L)$ converging to $u \in L^2(\Gamma;\psi)$ and suppose that the sequence $f_k = L_{\lambda_0} u_k$ converges to some f in $L^2(\Gamma;\psi)$. We have to prove that $L_{\lambda_0} u = f$. By definition each u_k is a weak solution of $L_{\lambda_0} u_k = f_k$. We may choose λ_0 large enough so that the bilinear form B_{λ_0} associated to L_{λ_0} is coercive. In this case we have $\|u_k\|_{H^1(\Gamma;\psi)} \le C\|f_k\|_{L^2(\Gamma;\psi)}$ (see Theorem 3.2.3) and from the regularity of weak solutions (see Proposition 3.2.6) we also have $\|u_k\|_{H^2(\Gamma)} \le C\|f_k\|_{L^2(\Gamma)}$. As f_k converges to f in $L^2(\Gamma;\psi)$ this implies that $(u_k)_{k\in\mathbb{N}}$ is bounded in $H^2(\Gamma;\psi)$. From the reflexivity of $H^2(\Gamma;\psi)$ we deduce that there exists a subsequence, which we still denote u_k, that converges weakly to some $\tilde{u} \in H^2(\Gamma;\psi)$. Now we use the fact that $B_{\lambda_0}(\cdot, v)$ belongs to $H^{-2}(\Gamma)$, the dual space of $H^2(\Gamma)$, for every $v \in H^2(\Gamma)$ to obtain

$$B(\tilde{u}, v) = (L\tilde{u}, v)_{L^2(\Gamma;\psi)} = \lim_{k\to\infty} (Lu_k, v)_{L^2(\Gamma;\psi)} = (f, v)_{L^2(\Gamma;\psi))}$$

for every $v \in H^2(\Gamma)$, which proves that $L\tilde{u} = f$. Finally as u_k converges to u in $L^2(\Gamma;\psi)$, we must have $\tilde{u} = u$. Thus we indeed have $L_{\lambda_0} u = f$. Finally as $\lambda_0 I$ is a bounded linear operator on $L^2(\Gamma;\psi)$ and $L = L\lambda_0 - \lambda_0 I$ we conclude that L is also a closed linear operator. $\square$

Proof A.4.3 (Proof of Theorem 4.2.2)**.** Consider $u_0 \in L^2(\Gamma)$ and $f \in L^2((0,T), L^2(\Gamma))$. We are going to use a regularization procedure. For this we introduce the following function

$$\rho \in \mathscr{C}_c^\infty(\mathbb{R}),\ 0 \le \rho,\ \operatorname{supp}(\rho) \subset (-1,1),\ \int_{\mathbb{R}} \rho(x)\, dx = 1$$

from which we define the family

$$\rho^\epsilon(x) = \frac{1}{\epsilon}\rho\left(\frac{x}{\epsilon}\right), \quad \forall \epsilon > 0.$$

Define also

$$\begin{gathered}\phi_\alpha^\epsilon \in \mathscr{C}_c^\infty((0,\ell_\alpha)),\ 0 \le \phi_\alpha^\epsilon \le 1,\\ \phi_\alpha^\epsilon(s) = 1\ \forall s \in [0,\ell_\alpha] \setminus ([0,2\epsilon) \cup (\ell_\alpha - 2\epsilon, \ell_\alpha]),\\ \phi_\alpha^\epsilon(s) = 0\ \forall s \in ([0,\epsilon) \cup (\ell_\alpha - \epsilon, \ell_\alpha]),\end{gathered}$$

for every $\alpha \in \mathcal{A}$. We then define the function $u_0^\epsilon : \Gamma \to \mathbb{R}$ by

$$u_{0,\alpha}^\epsilon(x) = \phi_\alpha^\epsilon(x)\left(\rho^\epsilon * \bar{u}_{0,\alpha}(x)\right), \quad \forall x \in [0,\ell_\alpha]$$

where $\bar{u}_{0,\alpha}$ is the extension by zero of $u_{0,\alpha}$ to $\mathbb{R}$. And

$$f^\epsilon(t,x) = \int_{\mathbb{R}} \rho(t-s,x)\tilde{f}(s,x)\, ds$$

where

$$\tilde{f}(t,x) = \begin{cases} f(t,x) & \text{if } 0 \le t \le T,\\ 0 & \text{otherwise.}\end{cases}$$

Lemma A.4.1. *For every $\epsilon > 0$ the function u_0^ϵ belongs to $\mathscr{C}^\infty(\Gamma) \cap D(L)$ and*

$$\lim_{\epsilon\to 0} \|u_0^\epsilon - u_0\|_{L^2(\Gamma)} = 0.$$

Moreover we also have $f^\epsilon \in \mathscr{C}^\infty([0,T], L^2(\Gamma))$ and

$$\lim_{\epsilon\to 0} \|f^\epsilon - f\|_{L^2((0,T),L^2(\Gamma))} = 0.$$

Proof. As $u_{0,\alpha} \in L^2(0,\ell_\alpha)$ we know that $\bar{u}_{0,\alpha} \in L^2(\mathbb{R})$ and it is then well-known that $\rho^\epsilon * \bar{u}_{0,\alpha} \in \mathscr{C}^\infty(\mathbb{R})$ (see [Bre11, Proposition 4.20]). Then because $\phi_\alpha^\epsilon \in \mathscr{C}_c^\infty((0,\ell_\alpha))$ we have that $u_{0,\alpha}^\epsilon \in \mathscr{C}^\infty([0,\ell_\alpha])$ and therefore $u_0^\epsilon \in \mathscr{C}^\infty(\Gamma)$. Moreover u_0^ϵ cancels on a neighborhood of every vertex, and thus it must satisfy the Kirchoff condition in $D(L)$. This proves the first statement.

For the second one notice that it is enough to prove that

$$\lim_{\epsilon\to 0} \|u_{0,\alpha}^\epsilon - u_{0,\alpha}\|_{L^2(0,\ell_\alpha)} = 0.$$

for every $\alpha \in \mathcal{A}$. First we have

$$\begin{aligned}\|u_{0,\alpha}^\epsilon - u_{0,\alpha}\|_{L^2(0,\ell_\alpha)} &\leq \|\phi_\alpha^\epsilon(\rho^\epsilon * u_{0,\alpha}^\epsilon - u_{0,\alpha}) + \phi_\alpha^\epsilon u_{0,\alpha}^\epsilon - u_{0,\alpha}\|_{L^2(0,\ell_\alpha)} \\ &\leq \|\rho^\epsilon * u_{0\,\alpha} - \bar{u}_{0,\alpha}\|_{L^2(\mathbb{R})} + \|\phi_\alpha^\epsilon u_{0,\alpha} - u_{0,\alpha}\|_{L^2(0,\ell_\alpha)}.\end{aligned}$$

because $\|\phi_\alpha^\epsilon\|_{L^\infty(0,\ell_\alpha)} \leq 1$. It is also well-known that (see [Bre11, Theorem 4.22] and [Bre11, p. 212])

$$\begin{aligned}\lim_{\epsilon\to 0} \|\rho^\epsilon u_{0\,\alpha} - \bar{u}_{0,\alpha}\|_{L^2(\mathbb{R})} &= 0, \\ \lim_{\epsilon\to 0} \|\phi_\alpha^\epsilon u_{0,\alpha} - u_{0,\alpha}\|_{L^2(0,\ell_\alpha)} &= 0.\end{aligned}$$

The result follows. The last statement also follows from [Bre11, Proposition 4.20, Proposition 4.22] noting that

$$\begin{aligned}\|f^\epsilon - f\|_{L^2((0,T),L^2(\Gamma))}^2 &= \int_0^T \int_\Gamma |f^\epsilon(t,x) - f(t,x)|^2 \, dxdt \\ &= \int_\Gamma \int_0^T |f^\epsilon(t,x) - f(t,x)|^2 \, dtdx.\end{aligned}$$

□

Using Lemma A.4.1 and Corollary 4.1.5 we find that there exists a unique semigroup solution

$$u^\epsilon \in \mathscr{C}^1((0,T), L^2(\Gamma)) \cap \mathscr{C}([0,T), H^2(\Gamma))$$

to

$$\text{(A.6)} \qquad \begin{cases} \partial_t u^\epsilon(t,x) + Lu^\epsilon(t,x) = f^\epsilon(t,x) & \text{for } t \in (0,T),\ x \in \Gamma \setminus \mathcal{V}, \\ \sum_{\alpha\in\mathcal{A}_i} \gamma_{i,\alpha}\sigma_{|\Gamma_\alpha}^2(v_i)\partial_\alpha u^\epsilon(t,v_i) = 0 & \forall i \in \mathcal{I} \text{ for every } t \in (0,T), \\ u_{|\Gamma_\alpha}^\epsilon(t,v_i) = u_{|\Gamma_\beta}^\epsilon(t,v_i) & \forall \alpha,\beta \in \mathcal{A}_i, \forall i \in \mathcal{I} \text{ for every } t \in (0,T), \\ u^\epsilon(0,\cdot) = u_0^\epsilon. \end{cases}$$

Moreover according to Theorem 4.1.6 we have

$$\begin{aligned}\|u^\epsilon\|_{L^2(0,T,H^1(\Gamma))} + \|\partial_t u^\epsilon\|_{L^2(0,T,H^{-1}(\Gamma))} &\le C\left(\|u_0^\epsilon\|_{L^2(\Gamma)} + \|f^\epsilon\|_{L^2((0,T),L^2(\Gamma))}\right)\\ &\le C\left(\|u_0\|_{L^2(\Gamma)} + \|f\|_{L^2((0,T),L^2(\Gamma))}\right)\end{aligned}$$

Using the reflexivity of $L^2(0,T,H^1(\Gamma))$ and $L^2(0,T,H^{-1}(\Gamma))$ we know that (see [Bre11, Theorem 3.18]) we find a sequence $(u^n)_{n\in\mathbb{N}}$ such that

$$\begin{cases} u^n \rightharpoonup u & \text{weakly in } L^2(0,T,H^1(\Gamma)),\\ \partial_t u^n \rightharpoonup v & \text{weakly in } L^2(0,T,H^{-1}(\Gamma)).\end{cases}$$

Then we have for $\phi \in \mathscr{C}_c^\infty((0,T),H^1(\Gamma))$

$$\begin{aligned}\int_0^T \langle u(t,\cdot),\partial_t\phi(t,\cdot)\rangle_{H^{-1},H^1}\,dt &= \lim_{n\to\infty}\int_0^T \langle u^n(t,\cdot),\partial_t\phi(t,\cdot)\rangle_{H^{-1},H^1}\,dt\\ &= \lim_{n\to\infty} -\int_0^T \langle \partial_t u^n(t,\cdot),\phi(t,\cdot)\rangle_{H^{-1},H^1}\,dt\\ &= -\int_0^T \langle v(t,\cdot),\phi(t,\cdot)\rangle_{H^{-1},H^1}\,dt\end{aligned}$$

which proves that $v=\partial_t u$ in the sense of distributions. Now recall from (4.7) that

$$(\partial_t u^n(t,\cdot),v)_{L^2(\Gamma;\psi)} + B(u^n(t,\cdot),v) = (f^n(t,\cdot),v)_{L^2(\Gamma;\psi)}$$

for every $v\in H^1(\Gamma;\psi)$ taking the weak limit (using also Lemma A.4.1 we find

$$(\partial_t u(t,\cdot),v)_{L^2(\Gamma;\psi)} + B(u(t,\cdot),v) = (f(t,\cdot),v)_{L^2(\Gamma;\psi)}$$

for every $v\in H^1(\Gamma;\psi)$.

Choosing now $v\in\mathscr{C}^1([0,T],H^1(\Gamma;\psi))$ such that $v(T,\cdot)=0$ and integrating by parts with respect to time we find that

$$\begin{aligned}&-\int_0^T (u^n(t,\cdot),\partial_t v(t,\cdot))_{L^2(\Gamma;\psi)} + B(u^n(t,\cdot),v(t,\cdot))\,dt\\ &\qquad= \int_0^T (f^n(t,\cdot),v(t,\cdot))_{L^2(\Gamma;\psi)}\,dt + (u^n(0,\cdot),v(0,\cdot))_{L^2(\Gamma;\psi)}\\ &\qquad= \int_0^T (f^n(t,\cdot),v(t,\cdot))_{L^2(\Gamma;\psi)}\,dt + (u_0^n,v(0,\cdot))_{L^2(\Gamma;\psi)}.\end{aligned}$$

Once again taking the weak limit we find that

$$\begin{aligned}&-\int_0^T (u(t,\cdot),\partial_t v(t,\cdot))_{L^2(\Gamma;\psi)} + B(u(t,\cdot),v(t,\cdot))\,dt\\ &\qquad= \int_0^T (f(t,\cdot),v(t,\cdot))_{L^2(\Gamma;\psi)}\,dt + (u_0,v(0,\cdot))_{L^2(\Gamma;\psi)}.\end{aligned}$$

Finally doing the same integration by parts directly on u gives

$$-\int_0^T (u(t,\cdot),\partial_t v(t,\cdot))_{L^2(\Gamma;\psi)} + B(u(t,\cdot),v(t,\cdot))\,dt$$
$$= \int_0^T (f(t,\cdot),v(t,\cdot))_{L^2(\Gamma;\psi)}\,dt + (u(0,\cdot),v(0,\cdot))_{L^2(\Gamma;\psi)}.$$

As v is arbitrary and comparing the two last equations we deduce that $u(0,\cdot) = u_0$ in $L^2(\Gamma)$. This proves that u is indeed a weak solution of the problem. □

Proof A.4.4 (Proof of Theorem 4.3.1)**.**

Lemma A.4.2. *There exists a positive constant λ_0 such that the bilinear form defined by*

$$\hat{B}_{\lambda_0}(u,v) = \hat{B}(u,v) + \lambda_0(u,v)_{\hat{L}^2(\Gamma;\psi)} \quad \forall (u,v) \in \hat{H}^1(\Gamma;\psi)\times\hat{H}^1(\Gamma;\psi)$$

satisfies

$$|\hat{B}_{\lambda_0}(u,u)| \geq C\|u\|_{\hat{H}^1(\Gamma;\psi)}.$$

Proof. First notice that $|B(u,u)| \geq \Re(B(u,u))$. Then as

$$\Re(B(u,u)) = \Re\left[\int_\Gamma a\partial u\partial(\bar{u}\psi) + \tilde{b}\partial u\bar{u}\psi + (c+\lambda_0)|u|\psi\,dx\right]$$
$$\geq \int_\Gamma \omega|\partial u|^2\psi - (\|a\partial\psi\|_{L^\infty(\Gamma)} + \|\tilde{b}\|_{L^\infty(\Gamma;\psi)})|u\partial u| + \lambda_0|u|^2\psi\,dx$$

we find that $\Re(B(u,u)) \geq C\|u\|_{\hat{H}^1(\Gamma,\psi)}$ for some positive constant C and λ_0 large enough using the same arguments as in Theorem 3.2.3. □

Lemma A.4.3. *Let λ_0 be such that the conclusion of Lemma A.4.2 holds. Then the operator*

$$-L - \lambda_0 I$$

satisfies $[0,+\infty) \subset \rho(-L-\lambda_0)$.

Proof. Let $\lambda \geq 0$. Then the bilinear form

$$\hat{B}_\lambda(u,v) = \hat{B}_{\lambda_0}(u,v) + \lambda(u,v)_{\hat{L}^2(\Gamma;\psi)}$$

is the bilinear form associated to the differential operator $\lambda I + L + \lambda_0 I$. According to Lemma A.4.2 it is coercive and hence applying the complex Lax-Milgram theorem ([Bre11, Proposition 11.29]) we find that $\lambda I + L + \lambda_0 I$ is invertible with continuous inverse, and hence $\lambda \in \rho(-L-\lambda_0 I)$. □

According to Lemma A.4.2 there exists a positive constant λ_0 such that

$$\Re\left[((L+\lambda_0 I)u,u)_{\hat{L}^2(\Gamma;\psi)}\right] = \Re\left[\hat{B}_{\lambda_0}(u,u)\right] \geq C\|u\|^2_{\hat{H}^1(\Gamma;\psi)} \geq C\|u\|^2_{\hat{L}^2(\Gamma;\psi)}$$

for every $u \in D(L)$ and some positive constant C. Notice also that

$$|\Im[((L+\lambda_0 I)u,u)_{\hat{L}^2(\Gamma;\psi)}]| \leq |\hat{B}_{\lambda_0}(u,u)| \leq C'\|u\|^2_{\hat{L}^2(\Gamma;\psi)}$$

for every $u \in D(L)$ and some positive constant C'. Choosing $u \in D(L)$ with $\|u\|_{\hat{L}^2(\Gamma;\psi)} = 1$ we find that there exists $0 < \theta < \frac{\pi}{2}$ such that the numerical range of $-L-\lambda_0 I$ (see Definition B.2.3 and Remark B.2.4) verifies

$$S(-L-\lambda_0 I) \subset S_\theta = \{\lambda \in \mathbb{C} :\ \pi - \theta < \arg\lambda < \pi + \theta\}\,.$$

Moreover we can also choose $\theta < \eta < \frac{\pi}{2}$ (see Fig. A.2) so that

$$\Sigma_\eta = \{\lambda \in \mathbb{C} :\ \pi - \eta < \arg\lambda < \pi + \eta\}^{\mathsf{c}}$$

and there exists a constant C_η such that

$$d(\lambda, \overline{S(-L-\lambda_0 I)}) \geq C_\eta|\lambda| \quad \forall \lambda \in \Sigma_\eta.$$

Furthermore from Lemma A.4.3 we know that $[0,+\infty) \subset \rho(-L-\lambda_0 I)$. This implies that $\rho(-L-\lambda_0 I) \cap \Sigma_\eta \neq \varnothing$ and by Proposition B.2.5 we have $\Sigma_\eta \subset \rho(-L-\lambda_0 I)$ and for every $\lambda \in \rho(-L-\lambda_0 I)$ we have

$$\|R_\lambda\|_{\mathscr{L}(\hat{L}^2(\Gamma;\psi))} \leq \frac{1}{d(\lambda, \overline{S(-L-\lambda_0 I)})} \leq \frac{1}{C_\eta|\lambda|}$$

We can now apply Theorem B.2.2 to obtain that $-L-\lambda_0 I$ is the infinitesimal generator of an analytic semigroup on $L^2(\Gamma;\psi)$. Finally as $\lambda_0 I$ is a bounded linear operator we deduce from Theorem B.2.6 that $-L = -L-\lambda_0 I + \lambda_0 I$ is also the infinitesimal generator of an analytic semigroup.

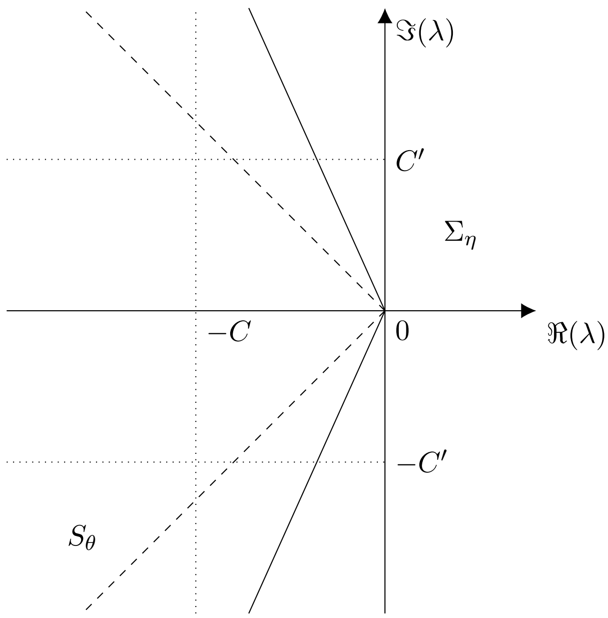

Figure A.2 – The sector S_θ is represented between the two dashed lines and the sector Σ_η between the two continuous lines.

□

A.5 Stochastic processes

Proof A.5.1 (Proof of Lemma 5.1.2)**.** Consider first the set

$$D = \left\{ u \in \mathscr{C}^2(\Gamma) : Lu \in \mathscr{C}(\Gamma),\ \partial u_\alpha(0) = \partial u_\alpha(\ell_\alpha) = 0 \text{ for every } \alpha \in \mathcal{A} \right\}$$

and notice that $D \subset D(L)$. Moreover if u is a constant function on Γ, then $u \in D$ and if $u, v \in D$, then $u + v \in D$. Furthermore we have for every $u, v \in D$ that

$$L(uv)(x) = u(x)Lv(x) + v(x)Lu(x) + 2\sigma^2(x)\partial u(x)\partial v(x)$$

which shows that $uv \in D$. It is also clear that the elements of D separate the points in Γ. Hence we can apply the Stone-Weierstrass theorem (see [Dix84, Theorem 7.5.3]) and we have that D is dense in $\mathscr{C}(\Gamma)$ which implies that $D(L)$ is dense in $\mathscr{C}(\Gamma)$. □

Proof A.5.2 (Proof of Lemma 5.1.3)**.** Let $(u_n)_{n\in\mathbb{N}}$ be a sequence in $D(L)$ such that u_n converges to some $u \in \mathscr{C}(\Gamma)$ and Lu_n converges to some f in $\mathscr{C}(\Gamma)$. According to Theorem 3.2.3 and Proposition 3.2.6 the operator $(\lambda I - L)$ is invertible with continuous inverse for $\lambda > 0$ and is bijective from $D(L)$ to $\mathscr{C}(\Gamma)$. Then we have

$$u_n = (\lambda I - L)^{-1}(\lambda u_n - Lu_n)$$

and taking the limit as n tends to infinity gives

$$u = (\lambda I - L)^{-1}(\lambda u - f)$$

which can be rewritten

$$\lambda u - Lu = \lambda u - f.$$

This implies that $Lu = f$ and therefore L is closed.

□

Proof A.5.3 (Proof of Lemma 5.1.4)**.** According to Theorem 3.2.9 and Proposition 3.2.6 we know that for every $f \in \mathscr{C}(\Gamma)$ there exists a unique $u_\lambda \in D(\Gamma)$ such that $(\lambda I - L)u_\lambda = f$ for every $\lambda > 0$. This proves the fist part of the lemma. Then we denote $(R_\lambda)_{\lambda>0}$ the resolvent that maps f to $(u_\lambda)_{\lambda>0}$. Now fix $\lambda > 0$ and denote $\bar{x} = \operatorname{argmax}_{x\in\Gamma} u_\lambda(x)$, $\underline{x} = \operatorname{argmin}_{x\in\Gamma} u_\lambda(x)$ and notice that

$$\|f\|_{\mathscr{C}(\Gamma)} \geq \max\{(\lambda I - L)u_\lambda(\bar{x}), -(\lambda I - L)u_\lambda(\underline{x})\}.$$

Moreover

$$(\lambda I - L)u_\lambda(\bar{x}) = \lambda u_\lambda(\bar{x}) - \sigma^2(\bar{x})\partial^2 u_\lambda(\bar{x}) \geq \lambda u_\lambda(\bar{x})$$

and

$$-(\lambda I - L)u_\lambda(\underline{x}) = -\lambda u_\lambda(\underline{x}) + \sigma^2(\underline{x})\partial^2 u_\lambda(\underline{x}) \geq -\lambda u_\lambda(\underline{x}),$$

is direct if $\bar{x}, \underline{x} \in \Gamma \setminus \mathcal{V}$. In the case where one of $\bar{x}$ or $\underline{x}$ is a vertex, we obtain the same result using Proposition 2.2.4. Therefore as $\|u_\lambda\|_{\mathscr{C}(\Gamma)} = \max\{u_\lambda(\bar{x}), -u_\lambda(\underline{x})\}$ we have obtained

$$\|u_\lambda\|_{\mathscr{C}(\Gamma)} \leq \frac{\|f\|_{\mathscr{C}(\Gamma)}}{\lambda},$$

that is

$$\|R_\lambda\|_{\mathscr{L}(\mathscr{C}(\Gamma))} \leq \frac{1}{\lambda}.$$

□

Appendix B

Semigroups of bounded linear operators

This section contains the basic facts about strongly continuous semigroups of bounded linear operators in Banach spaces and the statement of the Hille-Yosida theorem. We follow [Paz83] were the reader can find the proofs of the results presented here.

B.1 First definitions and the Hille-Yosida theorem

Definition B.1.1. A family $(T_t)_{t\geq 0}$ of bounded linear operator from Banach space X into itself is called a semigroup of bounded linear operators if it satisfies

1. $T_0 = I$,
2. for every $s, t \geq 0$ we have $T_{s+t} = T_s \circ T_t$.

Furthermore a semigroup $(T_t)_{t\geq 0}$ of bounded linear operators on X is said to be a strongly continuous semigroup, or a $\mathscr{C}_0$-semigroup, if

$$\lim_{t\to 0} \|T_t x - x\|_X = 0$$

for every $x \in X$.

Finally a semigroup $(T_t)_{t\geq 0}$ is said to be uniformly bounded if there exists a positive constant M such that

$$\|T_t\|_{\mathscr{L}(X)} \leq M$$

for every $t \geq 0$. ◁

Remark B.1.2. *For simplicity we will always write $T_s T_t$ for $T_s \circ T_t$.*

Definition B.1.3. Let $(T_t)_{t\geq 0}$ be a semigroup of bounded linear operator on X. We define the unbounded linear operator A on the domain

$$X \supset D(A) = \left\{ x \in X : \lim_{t\to 0} \frac{T_t x - x}{t} \text{ exists} \right\}$$

by

$$Ax = \lim_{t \to 0} \frac{T_t x - x}{t}, \quad x \in D(L).$$

The operator A is called the infinitesimal generator of a strongly continuous semigroup on X. ◁

Proposition B.1.4 ([Paz83], Theorem 1.2.4). *Let $(T_t)_{t\geq 0}$ be a strongly continuous semigroup and let L be its infinitesimal generator. Then for every $x \in D(A)$ we have $T_t x \in D(L)$, the mapping $t \mapsto T_t x$ is differentiable and*

$$\frac{d}{dt} T_t x = A T_t x = T_t A x,$$

Proposition B.1.5 ([Paz83], Theorem 1.2.6). *Let $(T_t)_{t\geq 0}$ and $(S_t)_{t\geq 0}$ be two strongly continuous semigroups of bounded linear operators sharing the same infinitesimal generator L with identical domain $D(L)$. Then $T_t = S_t$ for every $t \geq 0$.*

Definition B.1.6. A strongly continuous semigroup of bounded linear operators $(T_t)_{t\geq 0}$ is said to be a strongly continuous semigroup of contraction if $\|T_t\|_{\mathscr{L}(X)} \leq 1$ for every $t \geq 0$. ◁

Definition B.1.7. Let A be an unbounded linear operator on X. The resolvent set $\rho(A)$ of A is the set of all real numbers λ such that $(\lambda I - A)$ is invertible and $R_\lambda = (\lambda I - A)^{-1}$ is a bounded linear operator. Moreover the famillly of bounded linear operators $(R_\lambda)_{\lambda \in \rho(A)}$ is called the resolvent of A. ◁

We are now able to state the Hille-Yosida theorem.

Theorem B.1.8 (Hille-Yosida, Theorem 1.3.1 [Paz83]). *An unbounded linear operator A is the infinitesimal generator if a strongly continuous semigroup of contraction $(T_t)_{t\geq 0}$ if, and only if*

1. *L is closed and $D(A)$ is dense in X,*
2. *The resolvent set $\rho(A)$ contains $(0, \infty)$ and for every $\lambda > 0$ we have*

$$\|R_\lambda\|_{\mathscr{L}(X)} \leq \frac{1}{\lambda}.$$

Corollary B.1.9 ([Paz83], Corollary 1.3.8). *An unbounded linear operator A is the infinitesimal generator if a strongly continuous semigroup of $(T_t)_{t\geq 0}$ satisfying $\|T_t\|_{\mathscr{L}(X)} \leq e^{\omega t}$ if, and only if*

1. *A is closed and $D(A)$ is dense in X,*
2. *The resolvent set $\rho(A)$ contains (ω, ∞) and for every $\lambda > \omega$ we have*

$$\|R_\lambda\|_{\mathscr{L}(X)} \leq \frac{1}{\lambda - \omega}.$$

Theorem B.1.10 ([Paz83], Theorem 1.8.3). *Let $(T_t)_{t\geq 0}$ be a strongly continuous semigroup of contraction and let $(R_\lambda)_{\lambda > 0}$ be the resolvent associated to its infinitesimal generator. Then for every $x \in X$ we have*

$$T_t x = \lim_{n \to \infty} \left(\frac{n}{t} R_{n/t}\right)^n x,$$

and the limit is uniform in t on any bounded interval.

B.2 Analytic semigroups

Definition B.2.1. Let $\Delta = \{z \in \mathbb{C} : \ \theta_1 < \arg(z) < \theta_2, \ \theta_1 < 0 < \theta_2\}$ and for every $z \in \Delta$ let T_z be a bounded linear operator on X. The family $(T_z)_{z\in\Delta}$ is called an analytic semigroup in Δ if

1. $\Delta \ni z \mapsto T_z$ is analytic in Δ,

2. $T_0 = I$ and
$$\lim_{\substack{z\to 0\\ z\in\Delta}} T_z x = x$$
for every $x \in X$,

3. $T_{z_1+z_2} = T_{z_1}T_{z_2}$ for every $z_1, z_2 \in \Delta$.

A semigroup $(T_t)_{t\geq 0}$ will be called analytic if it can be extended to an analytic semigroup in some sector Δ containing $[0, +\infty)$. ◁

Theorem B.2.2 ([Paz83], Theorem 2.5.2)**.** *Let $(T_t)_{t\geq 0}$ be an uniformly bounded strongly continuous semigroup on X. Let A be its infinitesimal generator, assume $0 \in \rho(A)$ and denote $(R_\lambda)_{\lambda\in\rho(A)}$ the resolvent associated to A. Then the following statements are equivalent*

1. *the semigroup $(T_t)_{t\geq 0}$ can be extended to an analytic semigroup,*

2. *There exist constants $0 < \delta < \frac{\pi}{2}$ and $M > 0$ such that*
$$\rho(A) \supset \Sigma = \left\{\lambda \in \mathbb{C} : \ |\arg \lambda| < \frac{\pi}{2} + \delta\right\} \cup \{0\}$$
and
$$\|R_\lambda\|_{\mathscr{L}(X)} \leq \frac{M}{|\lambda|} \quad \text{for } \lambda \in \Sigma, \lambda \neq 0.$$

3. *$(T_t)_{t\geq 0}$ is differentiable for $T > 0$ and there is a constant C such that*
$$\|AT_t\|_{\mathscr{L}(x)} \leq \frac{C}{t}$$
for $t > 0$.

Definition B.2.3. The *numerical range* of an unbounded linear operator A is defined as

$$S(A) = \{\langle f, Ax\rangle_{X^\star,X} : \ x \in D(A), \ \|x\|_X = 1, \ \|f\|_{X^\star} = 1, \ \langle f, x\rangle_{X^\star,X} = 1\}.$$

◁

Remark B.2.4. *Let H be an Hilbert space and A an unbounded linear operator on H then we have*
$$S(A) = \{(Au, u)_H : \ u \in D(A), \ \|u\|_H = 1\}.$$

Proposition B.2.5 ([Paz83], Theorem 1.3.9)**.** *Let A be a closed linear operator with dense domain $D(A)$ in X. Let $S(A)$ be the numerical range of A and let Σ be the complement of $\overline{S(A)}$ in $\mathbb{C}$. If $\lambda \in \Sigma$ then $\lambda I - A$ invertible with continuous inverse. Moreover, if Σ_0 is a component of Σ satisfying $\rho(A) \cap \Sigma_0 \neq \varnothing$ then $\Sigma_0 \subset \rho(A)$ and the resolvent $(R_\lambda)_{\lambda \in \rho(A)}$ satisfies*

$$\|R_\lambda\|_{\mathscr{L}(X)} \leq \frac{1}{d(\lambda, \overline{S(A)})}.$$

Theorem B.2.6 ([Paz83], Corollary 3.2.2)**.** *Let A be the infinitesimal generator of an analytic semigroup. If B is a bounded linear operator on X then $A + B$ is also the infinitesimal generator of an analytic semigroup.*

B.3 Abstract Cauchy problems

In this section we want to consider the problem of finding a function $u : X \to X$ satisfying

$$\begin{cases} u'(t) = Au(t) + f(t) & \forall t > 0 \\ u(0) = u_0 \end{cases} \tag{B.1}$$

where A is an unbounded linear operator on X with domain $D(A) \subset X$ and $f : (0, +\infty) \to X$ is a given functions and $u_0 \in X$.

B.3.1 The homogeneous problem

We first give results concerning the following homogeneous problem

$$\begin{cases} u'(t) = Au(t) & \forall t > 0 \\ u(0) = u_0 \end{cases} \tag{B.2}$$

Theorem B.3.1 ([Paz83], Theorem 4.1.3)**.** *Assume A is the infinitesimal generator of a strongly continuous semigroup $(T_t)_{t \geq 0}$ on X. Then if $u_0 \in D(A)$ there is a unique solution*

$$u \in \mathscr{C}^1([0, \infty), X) \cap \mathscr{C}([0, \infty), D(A))$$

to the homogeneous problem (B.2) *given by $u(t) = T_t u_0$.*

Theorem B.3.2 ([Paz83], Corollary 4.1.5)**.** *Assume A is the infinitesimal generator of an analytic semigroup $(T_t)_{t \geq 0}$ on X. Then for every $u_0 \in X$ there is a unique solution*

$$u \in \mathscr{C}^1((0, \infty), X) \cap \mathscr{C}((0, \infty), D(A)) \cap \mathscr{C}([0, \infty), X)$$

to the homogeneous problem (B.2) *given by $u(t) = T_t u_0$.*

B.3.2 The inhomogeneous problem

We no turn to solutions of (B.1).

Theorem B.3.3 ([Paz83], Corollary 4.2.5)**.** *Assume A is the infinitesimal generator of a strongly continuous semigroup $(T_t)_{t\geq 0}$ on X. If $f \in \mathscr{C}^1([0,T],X)$ and $u_0 \in D(A)$ then* (B.1) *as a unique solution*

$$u \in \mathscr{C}^1((0,T),X) \cap \mathscr{C}([0,T),D(A))$$

given by

$$u(t) = T_t u_0 + \int_0^t T_{t-s} f(s)\, ds \quad \forall t \in [0,T).$$

Theorem B.3.4 ([Paz83] Corollary 4.3.3)**.** *Assume A is the infinitesimal generator of an analytic semigroup $(T_t)_{t\geq 0}$ on X. If $f \in W^{1,\infty}((0,T),X)$ then for every $u_0 \in X$ there exists a unique solution of* (B.1)

$$u \in \mathscr{C}^1((0,T),X) \cap \mathscr{C}((0,T),D(A)) \cap \mathscr{C}([0,T),X)$$

given by

$$u(t) = T_t u_0 + \int_0^t T_{t-s} f(s)\, ds \quad \forall t \in [0,T).$$

B.3.3 Semilinear problems

We now consider semilinear problems of the form

$$\begin{cases} u'(t) + Au(t) = f(t,u) & \forall t \in (0,T), \\ u(t_0) = u_0. \end{cases} \tag{B.3}$$

For this purpose we give some results on fractional powers of linear operator. We follow [Paz83, Section 2.2.6] were the reader can find the profs of the results stated here as well as a more in-depth presentation.

In what follows we assume that the unbounded linear operator A satisfies the following assumption.

Assumption 6. The operator A is a densely defined closed linear operator such that

$$\rho(A) \supset \Sigma^+ = \{\lambda \in \mathscr{C} :\ 0 < \omega < |\arg \lambda| \leq \pi\} \cup V$$

where V is a neighborhood of zero, and

$$\|R_\lambda\|_{\mathscr{L}(X)} \leq \frac{M}{1+|\lambda|} \quad \forall \lambda \in \Sigma^+.$$

$\triangleleft$

Let $\alpha \in (0,1)$, then on can define a linear operator A^α on a domain $D(A^\alpha) \supset D(A)$ (see [Paz83, Section 2.2.6] for the details of the construction). The following proposition summarizes the basic properties of this operator.

Proposition B.3.5. *Let $\alpha, \beta \in (0,1)$. Then*

1. *the operator A^α is closed and densely defined in X,*
2. *if $\alpha \geq \beta$ then $D(A^\alpha) \subset D(A^\beta)$,*
3. *If $\alpha + \beta < 1$ then*
$$A^{\alpha+\beta}x = A^\alpha A^\beta x$$
for every $x \in D(A^{\alpha+\beta})$ [1],
4. *If $x \in D(A)$ then*
$$A^\alpha x = \frac{\sin(\pi\alpha)}{\pi} \int_0^\infty t^{\alpha-1} A(tI + A)^{-1} x dt.$$

The following theorem will be important for the study of semilinear parabolic equations.

Theorem B.3.6. *Let B be a closed linear operator satisfying $D(B) \supset D(A)$. If for some $\gamma \in (0,1)$ and every $\eta \geq \eta_0 > 0$ we have*
$$\|Bx\|_X \leq C(\eta^\gamma \|x\|_X + \eta^{\gamma-1}\|Ax\|_X) \quad \forall x \in D(A),$$
then $D(A^\alpha) \subset D(B)$ for every $\gamma < \alpha \leq 1$.

Finally we define the Banach space X_α as the set $D(A^\alpha) \subset X$ provided with the graph norm of A^α.

We have to make an assumption on the function f.

Assumption 7. Let U be an open subset of $\mathbb{R}^+ \times X_\alpha$. The function $f : U \to X$ is such that for every $(t, x) \in U$ there is a neighborhood $V \subset U$ and constants $L \geq 0$, $0 < \theta \leq 1$ such that
$$\|f(t_1, x_1) - f(t_2, x_2)\|_X \leq L(|t_1 - t_2|^\theta + \|x_1 - x_2\|_{X_\alpha}) \quad \forall (t_i, x_i) \in V.$$
$\triangleleft$

We now state the main existence result for semilinear problems (see [Paz83, Theorem 6.3.1] or [Hen81, Theorem 3.3.3]).

Theorem B.3.7. *Let $-A$ be the infinitesimal generator of an analytic semigroup $(T_t)_{t\geq 0}$ satisfying $\|T_t\|_{\mathscr{L}(X)} \leq M$ and assume further that $0 \in \rho(-A)$. If $\alpha \in (0,1)$ and f satisfies Assumption 7 then for every initial data $(t_0, u_0) \in U$ the initial value problem*

$$\begin{cases} u'(t) + Au(t) = f(t,u) & \forall t \in (t_0, T), \\ u(t_0) = u_0. \end{cases} \tag{B.4}$$

has a unique solution
$$u \in \mathscr{C}([t_0, t_1), X) \cap \mathscr{C}^1((t_0, t_1), X) \cap \mathscr{C}((t_0, t_1), D(A))$$

[1] In fact one can define fractional powers of linear operators for any $\alpha \in \mathbb{R}$ and the result remains true in this case, see [Paz83, Section 2.2.6].

for some $t_1 > t_0$. *Furthermore we have for every* $t \in (t_0, t_1)$ *and*

$$u(t) = T_{t-t_0}u_0 + \int_{t_0}^{t} T_{t-s} f(s, u(s))\, ds.$$

Furthermore we under the right growth condition on f we can obtain a global existence and uniqueness result, see [Paz83, Theorem 6.3.3], [Hen81, Corollary 3.3.5]. The second statement is a consequence of [Hen81, Lemma 3.5.1]

Theorem B.3.8. *Let* $-A$ *be the infinitesimal generator of an analytic semigroup* $(T_t)_{t\geq 0}$ *satisfying* $\|T_t\|_{\mathscr{L}(X)} \leq M$ *and assume further that* $0 \in \rho(-A)$. *If* $\alpha \in (0,1)$ *and* $f : [t_0, +\infty) \times X_\alpha \to X$ *satisfies Assumption 7 and if there exists a continuous nondecreasing function* $k : [t_0, +\infty) \to \mathbb{R}_+$ *such that*

$$\|f(t,u)\|_X \leq k(t)(1 + \|u\|_{X_\alpha}) \quad \forall t \in [t_0, +\infty), \forall u \in X_\alpha.$$

Then for every $u_0 \in X_\alpha$ *the initial value problem*

$$\begin{cases} u'(t) + Au(t) = f(t,u) & \forall t \in (t_0, T), \\ u(t_0) = u_0. \end{cases} \tag{B.5}$$

has a unique solution for every $T > t_0$. *Moreover if* k *is a constant then* u' *is locally Hölder continuous with values in* X_γ *for* $0 < \theta < \gamma$ *for every* $t \in (0, T)$.

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